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Laws for third-order moments in homogeneous anisotropic incompressible magnetohydrodynamic turbulence

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It is known that Kolmogorov's four-fifths law for statistically homogeneous and isotropic turbulence can be generalized to anisotropic turbulence. This fundamental result for homogeneous anisotropic turbulence says that in the inertial range the divergence of the vector third-order moment $\langle |\delta \boldsymbol{v}(\boldsymbol{r})|^2 \delta \boldsymbol{v}(\boldsymbol{r})\rangle$ is constant and is equal to -4ε, where ε is the dissipation rate of the turbulence. This law can be extended to incompressible magnetohydrodyamic (MHD) turbulence where statistical isotropy is often not valid due, for example, to the presence of a large-scale magnetic field. Laws for anisotropic incompressible MHD turbulence were first derived by Politano and Pouquet. In this paper, the laws for vector third-order moments in homogeneous non-isotropic incompressible MHD turbulence are derived by a technique due to Frisch that clarifies the relationship between the energy flux in Fourier space and the vector third-order moments in physical space. This derivation is different from the original derivation of Politano and Pouquet which is based on the Kármán–Howarth equation, and provides some new physical insights. Separate laws are derived for the cascades of energy, cross-helicity and magnetic-helicity, the three ideal invariants of incompressible MHD for flows in three dimensions. These laws are of fundamental importance in the theory of MHD turbulence where non-isotropic turbulence is much more prevalent than isotropic turbulence.

1. Introduction

One of the few rigorous results in the theory of homogeneous isotropic turbulence for incompressible fluids is Kolmogorov's four-fifths law (Kolmogorov 1941), valid in the limit of large Reynolds number,

$$\langle [\delta v_{\parallel}(\boldsymbol{r})]^3 \rangle = -\tfrac{4}{5}\varepsilon r \tag{1.1}$$

where the length scale r lies in the inertial range, $\delta v_{\parallel}(\boldsymbol{r})$ is the component of the velocity fluctuation $\delta \boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x})$ in the direction of the displacement $\boldsymbol{r}$, ε is the average energy dissipation rate per unit mass, and angle brackets denote the ensemble average. Kolmogorov's four-fifths law tells us that in the inertial range, the third-order moment of the parallel velocity fluctuation $\delta v_{\parallel}$ is proportional to both the displacement r and the dissipation rate ε with a constant of proportionality that is determined by the theory to have the value $-4/5$.

Kolmogorov's four-fifths law is a special case of the more general law for homogeneous but not necessarily isotropic turbulence

$$\nabla \cdot \boldsymbol{F}(\boldsymbol{r}) = -4\varepsilon, \tag{1.2}$$

where

$$\boldsymbol{F}(\boldsymbol{r}) = \langle |\delta \boldsymbol{v}(\boldsymbol{r})|^2 \delta \boldsymbol{v}(\boldsymbol{r}) \rangle \tag{1.3}$$

and $|\boldsymbol{v}|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$. The quantity $\boldsymbol{F}(\boldsymbol{r})$ is called the vector third-order moment of the fluctuations. Equation (1.2) expresses the conservation of energy flux through the inertial range and shall be called the law for the inertial range energy flux. The law (1.2) is identical to equation (6.58) in Frisch (1995) and equation (22.15) on p. 402 of Monin & Yaglom (1975) where it is credited to Monin (1959). Derivations of this law have also been presented by Antonia *et al.* (1997) and by Hill (1997). It is important to emphasize that the divergence law (1.2) applies to both isotropic and anisotropic turbulence.

If the statistical properties of the turbulence are homogeneous and isotropic, then, by symmetry, the vector $\boldsymbol{F}(\boldsymbol{r})$ must have the form

$$\boldsymbol{F}(\boldsymbol{r}) = A\boldsymbol{r}, \tag{1.4}$$

where A is a scalar function of $\boldsymbol{r} \cdot \boldsymbol{r}$. The inertial range is described by the behaviour near $\boldsymbol{r} = 0$ where, because $\boldsymbol{F}$ is differentiable, A is constant. Therefore, the substitution of (1.4) into (1.2) yields the four-thirds law for homogeneous and isotropic turbulence

$$\boldsymbol{F}(\boldsymbol{r}) = -\tfrac{4}{3}\varepsilon \boldsymbol{r}. \tag{1.5}$$

This equation says that the statistical third-order moment $\boldsymbol{F}(\boldsymbol{r})$ is proportional to the displacement $\boldsymbol{r}$, like the four-fifths law (1.1), but it is a vector relation and, therefore, it contains information about both the parallel and perpendicular components of the fluctuations. In fact, it contains Kolmogorov's four-fifths law (1.1) as a special case.

The fundamental divergence law (1.2)–(1.3) can be extended from incompressible hydrodynamics to incompressible magnetohydrodynamics (MHD) in a straightforward manner as first shown by Politano & Pouquet (1998*a*). Politano, Gomez & Pouquet (2003) derived a similar law for the magnetic-helicity. Perhaps unwittingly, Politano & Pouquet (1998*a*) do not mention the important fact that their equation (3) is valid for *anisotropic* turbulence; but this fact is crucial since MHD turbulence is generally anisotropic as shown by laboratory experiments, theoretical studies (Montgomery & Turner 1981, 1982), numerical simulations (Shebalin, Matthaeus & Montgomery 1983; Oughton, Priest & Matthaeus 1994; Matthaeus *et al.* 1996*a*), and solar wind observations (Matthaeus, Bieber & Zank 1996*b*). Instead, Politano & Pouquet (1998*a*, *b*) immediately introduced the assumption of isotropic turbulence which obscured their more general result. It is important to emphasize that this more general result has great potential for applications to anisotropic plasma turbulence.

Because of the fundamental importance of the Politano & Pouquet laws for MHD turbulence, it is of interest to derive these laws in different ways. More than a theoretical exercise, this is useful to elucidate the physics and may possibly open the door to new lines of investigation. The derivation by Politano & Pouquet (1998*a*, *b*) is based on the anisotropic variant of the Kármán–Howarth equation for MHD and is similar to the derivation by Antonia *et al.* (1997) in the hydrodynamic case. In the present study, the MHD laws are derived by generalizing the method employed by Frisch (1995) from incompressible hydrodynamics to incompressible MHD.

Here, the MHD analogue of the divergence law (1.2) is derived in the limit $Re \to \infty$ and for arbitrary positive values of the magnetic Prandtl number $Pr_m = \nu/\eta$, where ν is the kinematic viscosity and η is the magnetic diffusivity. The principal results are equations (7.1)–(7.4). The derivation below uses the primitive variables $\boldsymbol{v}$ and $\boldsymbol{b}$ rather than the Elsasser variables $\boldsymbol{z}^{\pm}$ employed by Politano & Pouquet (1998*a*, *b*), a formulation that has certain advantages in applications that require tensor analysis (Podesta, Forman & Smith 2007). Following Frisch (1995), the derivation presented here is based on an equation for the scale-by-scale energy budget in Fourier space and an expression that relates the energy flux Π_K in $\boldsymbol{k}$-space to the third-order moments of the fluctuations in physical space. This relationship between the third-order moments in physical space and the energy flux in $\boldsymbol{k}$-space reveals the physical meaning of the fundamental law (1.2) in an especially simple way as discussed in § 7. Note that the law (7.3) for the cascade of magnetic-helicity in anisotropic turbulence is new; the derivation by Politano *et al.* (2003) is restricted to the case of isotropic turbulence. The laws (7.1) and (7.2) for the cascade of energy and cross-helicity in anisotropic turbulence were first derived by Politano & Pouquet (1998*a*).

In the present study, the derivation of the MHD laws for the third-order moments is composed of four principal elements or building blocks that are derived in the following four sections. The conservation of energy, cross-helicity and magnetic-helicity are derived in § 2. The scale-by-scale versions of these conservation laws are derived in § 3. The definitions of the third-order moments are derived in § 4. The spectral transfer rates are expressed in terms of the third-order moments in § 5. These four principal elements are then brought together in § 6 where the major results of this paper are developed. Even though the three laws (7.1), (7.2) and (7.3) are derived together, side by side, it should be emphasized that the derivation of each one is independent of the other two.

2. Conservation of energy, cross-helicity and magnetic-helicity

The governing equations of incompressible MHD are written

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} = -\nabla p + \nu\nabla^2\boldsymbol{v} + \boldsymbol{f}_v, \tag{2.1}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{v} = \eta\nabla^2\boldsymbol{b} + \boldsymbol{f}_b, \tag{2.2}$$

$$\nabla\cdot\boldsymbol{v} = \nabla\cdot\boldsymbol{b} = 0, \tag{2.3}$$

where $\boldsymbol{v}$ is the velocity, $\boldsymbol{b} = \boldsymbol{B}/\sqrt{\rho\mu_0}$ is the magnetic field in velocity units, $\rho =$ constant is the mass density, p is the total pressure (kinetic plus magnetic), ν is the kinematic viscosity, η is the magnetic diffusivity, $\boldsymbol{f}_v$ is the forcing function for the velocity field, $\boldsymbol{f}_b$ is the forcing function for the magnetic field, $\nabla\cdot\boldsymbol{f}_v = \nabla\cdot\boldsymbol{f}_b = 0$, and μ_0 is the permeability of free space (SI units). Note that the magnetic forcing term $\boldsymbol{f}_b$ in (2.2) is artificial because such a term never exists in real physical systems. It is sometimes used in numerical simulations of MHD turbulence to inject magnetic energy into the flow and is used here as a theoretical artifice.

For homogeneous incompressible MHD turbulence in three spatial dimensions the governing equations for the average energy, cross-helicity, and magnetic-helicity are

$$\frac{1}{2}\frac{\partial}{\partial t}\langle|\boldsymbol{v}|^2 + |\boldsymbol{b}|^2\rangle = -\nu\langle|\boldsymbol{\omega}|^2\rangle - \eta\langle|\boldsymbol{j}|^2\rangle + \langle\boldsymbol{v}\cdot\boldsymbol{f}_v + \boldsymbol{b}\cdot\boldsymbol{f}_b\rangle, \tag{2.4}$$

$$\frac{\partial}{\partial t}\langle\boldsymbol{v}\cdot\boldsymbol{b}\rangle = -(\nu+\eta)\langle\boldsymbol{j}\cdot\boldsymbol{\omega}\rangle + \langle\boldsymbol{v}\cdot\boldsymbol{f}_b + \boldsymbol{b}\cdot\boldsymbol{f}_v\rangle, \tag{2.5}$$

$$\frac{\partial}{\partial t}\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle = -2\eta\langle\boldsymbol{j}\cdot\boldsymbol{b}\rangle + \langle\boldsymbol{a}\cdot\boldsymbol{f}_b + \boldsymbol{b}\cdot\boldsymbol{f}_a\rangle, \tag{2.6}$$

respectively, where $\boldsymbol{\omega} = \nabla\times\boldsymbol{v}$ is the vorticity, $\boldsymbol{j} = \nabla\times\boldsymbol{b}$ is proportional to the electric current density, $\boldsymbol{a}$ is the vector potential for the magnetic field, $\boldsymbol{b} = \nabla\times\boldsymbol{a}$, $\boldsymbol{f}_b = \nabla\times\boldsymbol{f}_a$, and $|\boldsymbol{v}|^2 = \boldsymbol{v} \cdot \boldsymbol{v}$. Equation (2.6) applies only to turbulence in three-dimensional space. In two dimensions, the quantity $\langle a^2 \rangle$ is conserved instead of $\langle \boldsymbol{a} \cdot \boldsymbol{b} \rangle$ (Biskamp 2003).

2.1. *Definition of cascade rates*

Cascade rates can be defined for turbulence that is either statistically stationary in time or freely decaying. Under steady-state conditions, the statistical averages are independent of time and the conservation of energy (2.4) implies

$$\langle \boldsymbol{v} \cdot \boldsymbol{f}_v + \boldsymbol{b} \cdot \boldsymbol{f}_b \rangle = \nu \langle |\boldsymbol{\omega}|^2 \rangle + \eta \langle |\boldsymbol{j}|^2 \rangle \equiv \varepsilon. \tag{2.7}$$

This equation says that in the steady state, the average energy injection rate (left-hand side) is equal to the average energy dissipation rate (right-hand side). By definition, for a turbulent MHD system, this is equal to the average energy cascade rate ε from large to small scales.

Similarly, the conservation of cross-helicity (2.5) implies

$$\langle \boldsymbol{v} \cdot \boldsymbol{f}_b + \boldsymbol{b} \cdot \boldsymbol{f}_v \rangle = (\nu + \eta) \langle \boldsymbol{j} \cdot \boldsymbol{\omega} \rangle \equiv \varepsilon_C. \tag{2.8}$$

This equation says that at steady state, the average injection rate of cross-helicity (left-hand side) is equal to the average dissipation rate of cross-helicity (right-hand side). This defines the average cascade rate of cross-helicity ε_C. Note that the cross-helicity, unlike the energy, is not positive-definite. Both the cross-helicity and the cascade rate of cross-helicity can be positive, negative or zero. However, by the previous equation, the cascade rate has the same algebraic sign as the cross-helicity itself. The direction of the cross-helicity cascade is always from large to small scales (Biskamp 2003) independent of whether the value of the cross-helicity is positive or negative.

One of the goals of this paper is to derive laws for the flux of magnetic helicity through the inertial range similar to the laws for the flux of energy and cross-helicity. To do this, it is necessary to postulate the existence of an asymptotic steady state as has been assumed in (2.7) and (2.8). At steady state, the conservation of magnetic-helicity (2.6) implies

$$\langle \boldsymbol{a} \cdot \boldsymbol{f}_b + \boldsymbol{b} \cdot \boldsymbol{f}_a \rangle = 2\eta \langle \boldsymbol{j} \cdot \boldsymbol{b} \rangle \equiv \varepsilon_M. \tag{2.9}$$

According to this equation, at steady state, the average injection rate of magnetic-helicity (left-hand side) is equal to the average dissipation rate of magnetic-helicity (right-hand side). This defines the average cascade rate of magnetic-helicity ε_M. While it is well known that magnetic-helicity usually undergoes an inverse cascade to large scales (Meneguzzi, Frisch & Pouquet 1981; Biskamp 2003), there may be situations where the magnetic-helicity cascade process can reach a steady state, for example, in the case of periodic boundary conditions.

Cascade rates can also be defined for freely decaying turbulence under certain conditions. For homogeneous freely decaying turbulence, the forcing terms are zero and the conservation of energy (2.4) takes the form

$$-\frac{1}{2} \frac{\partial}{\partial t} \langle |\boldsymbol{v}|^2 + |\boldsymbol{b}|^2 \rangle = \nu \langle |\boldsymbol{\omega}|^2 \rangle + \eta \langle |\boldsymbol{j}|^2 \rangle \equiv \varepsilon(t). \tag{2.10}$$

This defines the time-dependent energy decay rate $\varepsilon(t)$. Similarly, the cross-helicity decay rate in freely decaying turbulence is defined by

$$-\frac{\partial}{\partial t} \langle \boldsymbol{v} \cdot \boldsymbol{b} \rangle = (\nu + \eta) \langle \boldsymbol{j} \cdot \boldsymbol{\omega} \rangle \equiv \varepsilon_C(t) \tag{2.11}$$

and the magnetic-helicity decay rate in freely decaying turbulence is defined by

$$-\frac{\partial}{\partial t}\langle \boldsymbol{a}\cdot\boldsymbol{b}\rangle = 2\eta\langle \boldsymbol{j}\cdot\boldsymbol{b}\rangle \equiv \varepsilon_M(t). \tag{2.12}$$

Note that the decay rates defined in (2.10)–(2.12) are not necessarily due to turbulent cascade processes. For example, the Taylor relaxation process in laboratory plasma experiments is characterized by a shortlived turbulent energy relaxation phase followed by a slowly decaying force-free state in which the magnetic-helicity decays gradually on a resistive time scale (Taylor 1986). The slow decay of the latter process is still described by the quantity $\varepsilon_M(t)$ defined in (2.12), even though the decay is not associated with a turbulent cascade process. In this study, attention is restricted to turbulent cascade processes. The decay rates $\varepsilon(t)$, $\varepsilon_C(t)$ and $\varepsilon_M(t)$ shall hereafter refer to turbulent cascade rates associated with nonlinear turbulent cascade processes (whenever they exist). It is not required, however, that in a given application all three cascade processes are simultaneously active. It is possible, for example, that there exists a turbulent energy cascade without a cascade of cross-helicity or magnetic-helicity so that $\varepsilon_C = \varepsilon_M = 0$ while $\varepsilon \neq 0$.

2.2. *Energy conservation*

The derivation of the energy conservation law (2.4) from the MHD equations is well known, but the technique is briefly reviewed because it is used frequently in later sections. Take the dot product of $\boldsymbol{v}$ with (2.1) plus the dot product of $\boldsymbol{b}$ with (2.2) to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\langle \boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{b}\cdot\boldsymbol{b}\rangle + NL = -\langle \boldsymbol{v}\cdot\nabla p\rangle + \nu\langle \boldsymbol{v}\cdot\nabla^2\boldsymbol{v}\rangle + \eta\langle \boldsymbol{b}\cdot\nabla^2\boldsymbol{b}\rangle + \langle \boldsymbol{v}\cdot\boldsymbol{f}_v + \boldsymbol{b}\cdot\boldsymbol{f}_b\rangle, \tag{2.13}$$

where the nonlinear terms are given by

$$NL = \langle \boldsymbol{v}\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \boldsymbol{v}\cdot(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} + \boldsymbol{b}\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - \boldsymbol{b}\cdot(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\rangle \tag{2.14}$$

and angle brackets denote a spatial average over a cube of size $L\times L\times L$. For homogeneous turbulence, the spatial and ensemble averages are equivalent. The limit $L\to\infty$ can be performed at the end of the calculation.

By incompressibility, the pressure term can be written

$$\langle \boldsymbol{v}\cdot\nabla p\rangle = \langle \nabla\cdot(p\boldsymbol{v})\rangle = \frac{1}{L^3}\int_S p\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{2.15}$$

where the surface integral on the right-hand side follows from the divergence theorem. If periodic boundary conditions are assumed, then the surface integral is zero. More generally, if all physical quantities p, $\boldsymbol{v}$, etc. are uniformly bounded in space and time, then the right-hand side vanishes in the limit $L\to\infty$.

After integration by parts, the viscous term becomes

$$\nu\langle \boldsymbol{v}\cdot\nabla^2\boldsymbol{v}\rangle = -\nu\langle |\nabla v_x|^2 + |\nabla v_y|^2 + |\nabla v_z|^2\rangle = -\nu\langle |\boldsymbol{\omega}|^2\rangle, \tag{2.16}$$

where the last equality on the right-hand side follows from the vector identity

$$\int \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})\,\mathrm{d}V = \int(\nabla\times\boldsymbol{A})\cdot\boldsymbol{B}\,\mathrm{d}V - \int(\boldsymbol{A}\times\boldsymbol{B})\cdot\boldsymbol{n}\,\mathrm{d}S \tag{2.17}$$

with $\boldsymbol{A} = \boldsymbol{v}$ and $\boldsymbol{B} = \nabla\times\boldsymbol{v}$. The contribution from the surface integral vanishes in the case of periodic boundary conditions or, as a consequence of uniform boundedness, after dividing by L^3 (to obtain the volume average) and then letting $L\to\infty$. Similar to the viscous term, the resistive term takes the form

$$\eta\langle \boldsymbol{b}\cdot\nabla^2\boldsymbol{b}\rangle = -\eta\langle|\nabla b_x|^2 + |\nabla b_y|^2 + |\nabla b_z|^2\rangle = -\eta\langle|\boldsymbol{j}|^2\rangle. \tag{2.18}$$

To complete the proof of (2.4) it only remains to show that the nonlinear term (2.14) vanishes. The first term in (2.14) can be written

$$\langle \boldsymbol{v}\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\rangle = \tfrac{1}{2}\langle(\boldsymbol{v}\cdot\nabla)(\boldsymbol{v}\cdot\boldsymbol{v})\rangle = \tfrac{1}{2}\langle\nabla\cdot[(\boldsymbol{v}\cdot\boldsymbol{v})\boldsymbol{v}]\rangle = \frac{1}{2L^3}\int_S(\boldsymbol{v}\cdot\boldsymbol{v})\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S, \tag{2.19}$$

where the second equality follows from incompressibility and the third from the divergence theorem. Uniform boundedness of $\boldsymbol{v}$ implies that the right-hand side vanishes as $L \rightarrow \infty$. The term $\langle\boldsymbol{b}\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{b}\rangle$ is evaluated in the same way. If the remaining two terms in (2.14) are combined, then they can also be evaluated in the same way. This completes the proof of (2.4).

2.3. *Cross-helicity conservation*

The derivation of (2.5) for the cross-helicity is similar to the derivation described in the preceding subsection. Take the dot product of $\boldsymbol{b}$ with (2.1) plus the dot product of $\boldsymbol{v}$ with (2.2) and then average. The remaining details are left to the reader. Note that as a consequence of incompressibility, the cross-helicity dissipation term can be written in the equivalent forms

$$\langle\boldsymbol{b}\cdot\nabla^2\boldsymbol{v}\rangle = -\langle\nabla v_x\cdot\nabla b_x + \nabla v_y\cdot\nabla b_y + \nabla v_z\cdot\nabla b_z\rangle = -\langle\boldsymbol{j}\cdot\boldsymbol{\omega}\rangle. \tag{2.20}$$

2.4. *Magnetic-helicity conservation*

The equations for the magnetic field and the magnetic vector potential may be written

$$\frac{\partial\boldsymbol{b}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{b}) + \eta\nabla^2\boldsymbol{b} + \boldsymbol{f}_b, \tag{2.21}$$

$$\frac{\partial\boldsymbol{a}}{\partial t} = \boldsymbol{v}\times\boldsymbol{b} + \eta\nabla^2\boldsymbol{a} + \boldsymbol{f}_a + \nabla\phi, \tag{2.22}$$

where $\boldsymbol{b} = \nabla\times\boldsymbol{a}$, $\boldsymbol{f}_b = \nabla\times\boldsymbol{f}_a$, and ϕ is the scalar potential. It can be assumed, without loss of generality, that $\nabla\cdot\boldsymbol{a} = 0$ since this can always be accomplished by an appropriate choice of ϕ. Take the dot product of $\boldsymbol{a}$ with (2.21) plus the dot product of $\boldsymbol{b}$ with (2.22) and then average to obtain

$$\frac{\partial}{\partial t}\langle\boldsymbol{a}\cdot\boldsymbol{b}\rangle = \langle\boldsymbol{a}\cdot[\nabla\times(\boldsymbol{v}\times\boldsymbol{b})]\rangle + \eta\langle\boldsymbol{a}\cdot\nabla^2\boldsymbol{b} + \boldsymbol{b}\cdot\nabla^2\boldsymbol{a}\rangle + \langle\boldsymbol{a}\cdot\boldsymbol{f}_b + \boldsymbol{b}\cdot\boldsymbol{f}_a\rangle + \langle\boldsymbol{b}\cdot\nabla\phi\rangle. \tag{2.23}$$

Using the vector identity (2.17), it can be shown that the first term on the right-hand side vanishes using the same kind of boundedness argument given for (2.15). The dissipative terms may also be evaluated using the vector identity (2.17) with the result

$$\eta\langle\boldsymbol{a}\cdot\nabla^2\boldsymbol{b} + \boldsymbol{b}\cdot\nabla^2\boldsymbol{a}\rangle = -2\eta\langle\boldsymbol{j}\cdot\boldsymbol{b}\rangle. \tag{2.24}$$

Because $\nabla\cdot\boldsymbol{b} = 0$, the term containing the scalar potential ϕ can be converted to a surface integral which vanishes by the same boundedness argument as given for (2.15). This completes the derivation of (2.6).

3. Scale-by-scale energy, cross-helicity and magnetic-helicity budget

For any wavenumber $K > 0$, the velocity field $\boldsymbol{v}(\boldsymbol{x})$ can be decomposed into two components, a low-wavenumber component containing all Fourier wavevectors of magnitude less than or equal to K, and a high-wavenumber component containing all wavevectors greater than K. In this section, an equation is derived for the conservation of energy for the low-wavenumber component. This equation contains terms for the energy generation rate due to forcing and the average dissipation rate, like (2.4), but it also contains a term describing the energy transfer from the low-wavenumber component to the high-wavenumber component. The latter term is the key to the entire theory developed in this paper, however, a useful mathematical form for this term cannot be derived until § 5.

For a domain with periodic boundary conditions, the Fourier series representation of $\boldsymbol{v}(\boldsymbol{x})$ is written

$$\boldsymbol{v}(\boldsymbol{x}) = \sum_{\boldsymbol{k}} \hat{\boldsymbol{v}}(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}), \tag{3.1}$$

where $\boldsymbol{k} = 2\pi \boldsymbol{n}/L$ with $\boldsymbol{n} = (n_1, n_2, n_3)$ an ordered triple of integers, and

$$\hat{\boldsymbol{v}}(\boldsymbol{k}) = \frac{1}{L^3} \int_{cube} \boldsymbol{v}(\boldsymbol{x}) \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}) \, \mathrm{d}^3\boldsymbol{x}, \tag{3.2}$$

where the integration is over a cube of volume L^3. Any dependent variable, such as $\boldsymbol{v}(\boldsymbol{x})$, can be decomposed into low- and high-wavenumber components in the form

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}_K^<(\boldsymbol{x}) + \boldsymbol{v}_K^>(\boldsymbol{x}), \tag{3.3}$$

where

$$\boldsymbol{v}_K^<(\boldsymbol{x}) = \sum_{|\boldsymbol{k}| \leqslant K} \hat{\boldsymbol{v}}(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}), \tag{3.4}$$

$$\boldsymbol{v}_K^>(\boldsymbol{x}) = \sum_{|\boldsymbol{k}| > K} \hat{\boldsymbol{v}}(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{x}). \tag{3.5}$$

3.1. *Energy budget scale-by-scale*

For homogeneous incompressible MHD turbulence, the conservation of energy for the low-wavenumber components of the fields takes the form

$$\frac{\partial E_K}{\partial t} = S_K - D_K - \Pi_K, \tag{3.6}$$

where

$$E_K = \tfrac{1}{2}\langle |\boldsymbol{v}_K^<|^2 + |\boldsymbol{b}_K^<|^2 \rangle \tag{3.7}$$

is the average energy of the low-wavenumber components (kinetic plus magnetic),

$$S_K = \langle \boldsymbol{v}_K^< \cdot \boldsymbol{f}_{vK}^< + \boldsymbol{b}_K^< \cdot \boldsymbol{f}_{bK}^< \rangle \tag{3.8}$$

is the average rate of energy injection at low wavenumbers due to forcing,

$$D_K = \nu \langle |\boldsymbol{\omega}_K^<|^2 \rangle + \eta \langle |\boldsymbol{j}_K^<|^2 \rangle \tag{3.9}$$

is the average energy dissipation rate at low wavenumbers due to viscous and resistive dissipation, and Π_K is the rate of energy transfer from low wavenumbers to high wavenumbers (from wavenumbers less than or equal to K to wavenumbers greater than K). The energy transfer rate Π_K arises from the nonlinear terms in the equations of motion. An expression for the energy transfer rate Π_K is given below.

Equation (3.6) is derived as follows. Let P_K denote the projection operator that projects out the low-wavenumber component of any function:

$$P_K \boldsymbol{v}(\boldsymbol{x}) = P_K[\boldsymbol{v}_K^<(\boldsymbol{x}) + \boldsymbol{v}_K^>(\boldsymbol{x})] = \boldsymbol{v}_K^<(\boldsymbol{x}). \tag{3.10}$$

Note that $P_K \boldsymbol{v}_K^<(\boldsymbol{x}) = \boldsymbol{v}_K^<(\boldsymbol{x})$. Apply P_K to (2.1) and (2.2) to obtain

$$\frac{\partial \boldsymbol{v}_K^<}{\partial t} + P_K[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}] = -\nabla p_K^< + \nu\nabla^2\boldsymbol{v}_K^< + \boldsymbol{f}_{vK}^<, \tag{3.11}$$

$$\frac{\partial \boldsymbol{b}_K^<}{\partial t} + P_K[(\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{v}] = \eta\nabla^2\boldsymbol{b}_K^< + \boldsymbol{f}_{bK}^<, \tag{3.12}$$

Take the dot product of $\boldsymbol{v}_K^<(\boldsymbol{x})$ with (3.11) plus the dot product of $\boldsymbol{b}_K^<(\boldsymbol{x})$ with (3.12) and then average to obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\langle|\boldsymbol{v}_K^<|^2 + |\boldsymbol{b}_K^<|^2\rangle + NL_1 = -\langle\boldsymbol{v}_K^<\cdot\nabla p_K^<\rangle + \nu\langle\boldsymbol{v}_K^<\cdot\nabla^2\boldsymbol{v}_K^<\rangle$$
$$+ \eta\langle\boldsymbol{b}_K^<\cdot\nabla^2\boldsymbol{b}_K^<\rangle + \langle\boldsymbol{v}_K^<\cdot\boldsymbol{f}_{vK}^< + \boldsymbol{b}_K^<\cdot\boldsymbol{f}_{bK}^<\rangle, \tag{3.13}$$

where the nonlinear terms are given by

$$NL_1 = \langle\boldsymbol{v}_K^<\cdot P_K[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}] + \boldsymbol{b}_K^<\cdot P_K[(\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{v}]\rangle \tag{3.14}$$

and angle brackets denote a spatial average over the cube of size $L\times L\times L$. Using the incompressibility condition $\nabla\cdot\boldsymbol{v}_K^< = 0$, it can be shown that the pressure term in (3.13) vanishes as in (2.15). The viscous and resistive terms can be evaluated using the vector identity (2.17).

The verification of (3.6)–(3.9) is completed by equating the energy transfer rate Π_K with the nonlinear term (3.14). This is consistent with the well-known fact that the nonlinear terms in the MHD equations cannot create or dissipate energy, but only redistribute energy in Fourier space. Because an alternative expression for this term is derived in §§4 and 5, the derivation in the remainder of this subsection can be skipped without loss of continuity. To evaluate the nonlinear term (3.14), observe that for any two functions f and g, $\langle f(P_K g)\rangle = \langle(P_K f)g\rangle$, so that the nonlinear term (3.14) is equal to

$$NL_1 = \langle\boldsymbol{v}_K^<\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}] + \boldsymbol{b}_K^<\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{v}]\rangle. \tag{3.15}$$

Expressing $\boldsymbol{v}$ and $\boldsymbol{b}$ in terms of high- and low-wavenumber components yields sixteen terms, eight of which can be shown to vanish. Thus, (3.15) yields the expression

$$\Pi_K = \langle\boldsymbol{v}_K^<\cdot[(\boldsymbol{v}_K^<\cdot\nabla)\boldsymbol{v}_K^> + (\boldsymbol{v}_K^>\cdot\nabla)\boldsymbol{v}_K^>]\rangle + \langle\boldsymbol{b}_K^<\cdot[(\boldsymbol{v}_K^<\cdot\nabla)\boldsymbol{b}_K^> + (\boldsymbol{v}_K^>\cdot\nabla)\boldsymbol{b}_K^>]\rangle$$
$$- \langle\boldsymbol{b}_K^<\cdot[(\boldsymbol{b}_K^<\cdot\nabla)\boldsymbol{v}_K^> + (\boldsymbol{b}_K^>\cdot\nabla)\boldsymbol{v}_K^>]\rangle - \langle\boldsymbol{v}_K^<\cdot[(\boldsymbol{b}_K^<\cdot\nabla)\boldsymbol{b}_K^> + (\boldsymbol{b}_K^>\cdot\nabla)\boldsymbol{b}_K^>]\rangle. \tag{3.16}$$

Eight of the original sixteen terms vanish as $L\to\infty$ due to identities of the form

$$\langle\boldsymbol{v}_K^<\cdot(\boldsymbol{v}_K^>\cdot\nabla)\boldsymbol{v}_K^<\rangle = \tfrac{1}{2}\langle(\boldsymbol{v}_K^>\cdot\nabla)(\boldsymbol{v}_K^<\cdot\boldsymbol{v}_K^<)\rangle = \tfrac{1}{2}\langle\nabla\cdot[(\boldsymbol{v}_K^<\cdot\boldsymbol{v}_K^<)\boldsymbol{v}_K^>]\rangle \to 0. \tag{3.17}$$

3.2. *Cross-helicity budget scale-by-scale*

For homogeneous incompressible MHD turbulence, the conservation of cross-helicity for the low wavenumber components of the fields takes the form

$$\frac{\partial H_K^C}{\partial t} = S_K^C - D_K^C - \Pi_K^C, \tag{3.18}$$

where

$$H_K^C = \langle\boldsymbol{v}_K^<\cdot\boldsymbol{b}_K^<\rangle \tag{3.19}$$

is the average cross-helicity of the low-wavenumber component,

$$S_K^C = \langle\boldsymbol{v}_K^<\cdot\boldsymbol{f}_{bK}^< + \boldsymbol{b}_K^<\cdot\boldsymbol{f}_{vK}^<\rangle \tag{3.20}$$

is the rate of cross-helicity injection at low wavenumbers due to forcing,

$$D_K^C = (\nu + \eta)\langle \boldsymbol{j}_K^< \cdot \boldsymbol{\omega}_K^< \rangle \tag{3.21}$$

is the average cross-helicity dissipation rate at low wavenumbers due to viscous and resistive dissipation, and Π_K^C is the rate of transfer of cross-helicity from low wavenumbers to high wavenumbers. An expression for the cross-helicity transfer rate Π_K^C is given below.

The derivation of (3.18)–(3.21) is similar to the derivation of (3.6)–(3.9). Apply the projection operator P_K to (2.1) and (2.2), then take the dot product of $\boldsymbol{b}_K^<(\boldsymbol{x})$ with the result of P_K applied to (2.1) and take the dot product of $\boldsymbol{v}_K^<(\boldsymbol{x})$ with the result of P_K applied to (2.2), add the two equations and then average to obtain

$$\frac{\partial}{\partial t}\langle \boldsymbol{v}_K^< \cdot \boldsymbol{b}_K^< \rangle + NL_2 = -\langle \boldsymbol{b}_K^< \cdot \nabla p_K^< \rangle + \nu \langle \boldsymbol{b}_K^< \cdot \nabla^2 \boldsymbol{v}_K^< \rangle + \eta \langle \boldsymbol{v}_K^< \cdot \nabla^2 \boldsymbol{b}_K^< \rangle + \langle \boldsymbol{b}_K^< \cdot \boldsymbol{f}_{vK}^< + \boldsymbol{v}_K^< \cdot \boldsymbol{f}_{bK}^< \rangle, \tag{3.22}$$

where

$$NL_2 = \langle \boldsymbol{b}_K^< \cdot P_K[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}] + \boldsymbol{v}_K^< \cdot P_K[(\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{v}]\rangle \tag{3.23}$$

and angle brackets denote a spatial average over a cube of size $L \times L \times L$. From the divergence-free condition $\nabla \cdot \boldsymbol{b}_K^< = 0$, the pressure term in (3.22) can be reduced to a surface integral that vanishes as $L \to \infty$ by the boundedness argument used in (2.15). The viscous and resistive terms can be evaluated using the vector identity (2.17) with the result

$$\nu \langle \boldsymbol{b}_K^< \cdot \nabla^2 \boldsymbol{v}_K^< \rangle + \eta \langle \boldsymbol{v}_K^< \cdot \nabla^2 \boldsymbol{b}_K^< \rangle = -(\nu + \eta)\langle \boldsymbol{j}_K^< \cdot \boldsymbol{\omega}_K^< \rangle. \tag{3.24}$$

It only remains to evaluate the nonlinear term (3.23) which is equal to the cross-helicity transfer rate Π_K^C. Because an alternative expression for this term is derived in §§4 and 5, the derivation in the remainder of this subsection can be skipped without loss of continuity. Moving the operator P_K from the term in square brackets to the term on the left-hand side of the dot product, the nonlinear term (3.23) can be written

$$NL_2 = \langle \boldsymbol{b}_K^< \cdot [(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}] + \boldsymbol{v}_K^< \cdot [(\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{v}]\rangle. \tag{3.25}$$

Expressing $\boldsymbol{v}$ and $\boldsymbol{b}$ in terms of high- and low-wavenumber components yields sixteen terms, eight of which can be shown to vanish. Thus, (3.25) becomes

$$\Pi_K^C = \langle \boldsymbol{b}_K^< \cdot [(\boldsymbol{v}_K^< \cdot \nabla)\boldsymbol{v}_K^> + (\boldsymbol{v}_K^> \cdot \nabla)\boldsymbol{v}_K^>]\rangle - \langle \boldsymbol{b}_K^< \cdot [(\boldsymbol{b}_K^< \cdot \nabla)\boldsymbol{b}_K^> + (\boldsymbol{b}_K^> \cdot \nabla)\boldsymbol{b}_K^>]\rangle + \langle \boldsymbol{v}_K^< \cdot [(\boldsymbol{v}_K^< \cdot \nabla)\boldsymbol{b}_K^> + (\boldsymbol{v}_K^> \cdot \nabla)\boldsymbol{b}_K^>]\rangle - \langle \boldsymbol{v}_K^< \cdot [(\boldsymbol{b}_K^< \cdot \nabla)\boldsymbol{v}_K^> + (\boldsymbol{b}_K^> \cdot \nabla)\boldsymbol{v}_K^>]\rangle. \tag{3.26}$$

Eight of the original sixteen terms vanish as $L \to \infty$ due to relations of the form

$$\langle \boldsymbol{b}_K^< \cdot (\boldsymbol{v}_K^< \cdot \nabla)\boldsymbol{v}_K^< + \boldsymbol{v}_K^< \cdot (\boldsymbol{v}_K^< \cdot \nabla)\boldsymbol{b}_K^< \rangle = \langle (\boldsymbol{v}_K^< \cdot \nabla)(\boldsymbol{v}_K^< \cdot \boldsymbol{b}_K^<)\rangle = \langle \nabla \cdot [(\boldsymbol{v}_K^< \cdot \boldsymbol{b}_K^<)\boldsymbol{v}_K^<]\rangle \to 0. \tag{3.27}$$

This completes the derivation of (3.18)–(3.21).

3.3. *Magnetic-helicity budget scale-by-scale*

For homogeneous incompressible MHD turbulence, the conservation of magnetic-helicity for the low wavenumber components of the fields takes the form

$$\frac{\partial H_K^M}{\partial t} = S_K^M - D_K^M - \Pi_K^M, \tag{3.28}$$

where

$$H_K^M = \langle \boldsymbol{a}_K^< \cdot \boldsymbol{b}_K^< \rangle \tag{3.29}$$

is the average magnetic-helicity of the low-wavenumber component,

$$S_K^M = \langle \boldsymbol{a}_K^< \cdot \boldsymbol{f}_{bK}^< + \boldsymbol{b}_K^< \cdot \boldsymbol{f}_{aK}^< \rangle \tag{3.30}$$

is the rate of magnetic-helicity injection at low wavenumbers due to forcing,

$$D_K^M = 2\eta \langle \boldsymbol{j}_K^< \cdot \boldsymbol{b}_K^< \rangle \tag{3.31}$$

is the average magnetic-helicity dissipation rate at low wavenumbers due to viscous and resistive dissipation, and Π_K^M is the rate of transfer of magnetic-helicity from low wavenumbers to high wavenumbers. An expression for the magnetic-helicity transfer rate Π_K^M is given below.

Equations (3.28)–(3.31) are derived as follows. Apply the projection operator P_K to (2.21) and (2.22), take the dot product of $\boldsymbol{a}_K^<(\boldsymbol{x})$ with the result of P_K applied to (2.21) and take the dot product of $\boldsymbol{b}_K^<(\boldsymbol{x})$ with the result of P_K applied to (2.22), then add the two equations and average to obtain

$$\frac{\partial}{\partial t}\langle \boldsymbol{a}_K^< \cdot \boldsymbol{b}_K^< \rangle = NL_3 + \eta \langle \boldsymbol{a}_K^< \cdot \nabla^2 \boldsymbol{b}_K^< + \boldsymbol{b}_K^< \cdot \nabla^2 \boldsymbol{a}_K^< \rangle + \langle \boldsymbol{a}_K^< \cdot \boldsymbol{f}_{bK}^< + \boldsymbol{b}_K^< \cdot \boldsymbol{f}_{aK}^< \rangle, \tag{3.32}$$

where

$$NL_3 = \langle \boldsymbol{a}_K^< \cdot P_K[\nabla \times (\boldsymbol{v} \times \boldsymbol{b})] + \boldsymbol{b}_K^< \cdot P_K(\boldsymbol{v} \times \boldsymbol{b}) \rangle \tag{3.33}$$

and angle brackets denote a spatial average over a cube of size $L \times L \times L$. The term on the left-hand side of (3.32) is equal to $\partial H_K^M / \partial t$. The third term on the right-hand side of (3.32) is S_K^M. The second term on the right-hand side of (3.32) can be evaluated using the vector identity (2.17) together with the fact that $\nabla \cdot \boldsymbol{a}_K^< = 0$. Hence,

$$\eta \langle \boldsymbol{a}_K^< \cdot \nabla^2 \boldsymbol{b}_K^< + \boldsymbol{b}_K^< \cdot \nabla^2 \boldsymbol{a}_K^< \rangle = -2\eta \langle \boldsymbol{j}_K^< \cdot \boldsymbol{b}_K^< \rangle \tag{3.34}$$

and this is equal to $-D_K^M$.

The nonlinear term (3.33) is equal to the magnetic-helicity transfer rate Π_K^M. Because an alternative expression for this term is derived in §§4 and 5, the derivation in the remainder of this subsection can be skipped without loss of continuity. Moving the operator P_K from the second term in the dot product to the first term in the dot product, the nonlinear term (3.33) takes the form

$$NL_3 = \langle \boldsymbol{a}_K^< \cdot [\nabla \times (\boldsymbol{v} \times \boldsymbol{b})] + \boldsymbol{b}_K^< \cdot (\boldsymbol{v} \times \boldsymbol{b}) \rangle. \tag{3.35}$$

The first term can be simplified using the vector identity (2.17) with the result

$$NL_3 = 2\langle \boldsymbol{b}_K^< \cdot (\boldsymbol{v} \times \boldsymbol{b}) \rangle, \tag{3.36}$$

where the surface term has been dropped since it vanishes in the limit as $L \to \infty$. Expressing $\boldsymbol{v}$ and $\boldsymbol{b}$ in terms of low- and high-wavenumber components, it follows that

$$\Pi_K^M = -NL_3 = -2\langle \boldsymbol{b}_K^< \cdot (\boldsymbol{v}_K^< \times \boldsymbol{b}_K^> + \boldsymbol{v}_K^> \times \boldsymbol{b}_K^>) \rangle. \tag{3.37}$$

This completes the derivation of (3.28)–(3.31).

4. Third-order moments $\boldsymbol{F}$, $\boldsymbol{F}^C$ and $\boldsymbol{F}^M$

The vectors $\boldsymbol{F}$, $\boldsymbol{F}^C$ and $\boldsymbol{F}^M$ which enter the divergence laws derived in §6 are third-order moments of the fields $\boldsymbol{v}$ and $\boldsymbol{b}$. These third-order moments originate in the dynamic equations for second-order correlation functions. In this section, equations for the second-order correlation functions are derived; the vectors $\boldsymbol{F}$, $\boldsymbol{F}^C$ and $\boldsymbol{F}^M$ which emerge from these equations are defined; and the divergence of these vectors are related to the time rate of change of the second-order correlation functions due to nonlinear interactions. The latter relationships are required in § 5.

4.1. *Third-order moment $\boldsymbol{F}$ for the energy*

To derive an equation for the two-point correlation function, write down equations for $\boldsymbol{v}$ and $\boldsymbol{b}$ at the two points $\boldsymbol{x}$ and $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b} = -\nabla p + \nu\nabla^2\boldsymbol{v} + \boldsymbol{f}_v, \tag{4.1}$$

$$\frac{\partial \boldsymbol{b}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{v} = \eta\nabla^2\boldsymbol{b} + \boldsymbol{f}_b; \tag{4.2}$$

$$\frac{\partial \boldsymbol{v}'}{\partial t} + (\boldsymbol{v}'\cdot\nabla)\boldsymbol{v}' - (\boldsymbol{b}'\cdot\nabla)\boldsymbol{b}' = -\nabla p' + \nu\nabla^2\boldsymbol{v}' + \boldsymbol{f}'_v, \tag{4.3}$$

$$\frac{\partial \boldsymbol{b}'}{\partial t} + (\boldsymbol{v}'\cdot\nabla)\boldsymbol{b}' - (\boldsymbol{b}'\cdot\nabla)\boldsymbol{v}' = \eta\nabla^2\boldsymbol{b}' + \boldsymbol{f}'_b, \tag{4.4}$$

where $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{v}' = \boldsymbol{v}(\boldsymbol{x} + \boldsymbol{r}, t)$, etc., and all spatial derivatives are with respect to the variable $\boldsymbol{x}$. Take the dot product of $\boldsymbol{v}'$ with (4.1), plus the dot product of $\boldsymbol{v}$ with (4.3), plus the dot product of $\boldsymbol{b}'$ with (4.2), plus the dot product of $\boldsymbol{b}$ with (4.4), and then average to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\langle \boldsymbol{v}'\cdot\boldsymbol{v} + \boldsymbol{b}'\cdot\boldsymbol{b}\rangle + NL = {} & -\langle \boldsymbol{v}'\cdot\nabla p + \boldsymbol{v}\cdot\nabla p'\rangle + \nu\langle \boldsymbol{v}'\cdot\nabla^2\boldsymbol{v} + \boldsymbol{v}\cdot\nabla^2\boldsymbol{v}'\rangle \\ & + \eta\langle \boldsymbol{b}'\cdot\nabla^2\boldsymbol{b} + \boldsymbol{b}\cdot\nabla^2\boldsymbol{b}'\rangle + \langle \boldsymbol{v}'\cdot\boldsymbol{f}_v + \boldsymbol{v}\cdot\boldsymbol{f}'_v + \boldsymbol{b}'\cdot\boldsymbol{f}_b + \boldsymbol{b}\cdot\boldsymbol{f}'_b\rangle,\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}NL = {} & \langle \boldsymbol{v}'\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}] + \boldsymbol{v}\cdot[(\boldsymbol{v}'\cdot\nabla)\boldsymbol{v}' - (\boldsymbol{b}'\cdot\nabla)\boldsymbol{b}'] \\ & + \boldsymbol{b}'\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{v}] + \boldsymbol{b}\cdot[(\boldsymbol{v}'\cdot\nabla)\boldsymbol{b}' - (\boldsymbol{b}'\cdot\nabla)\boldsymbol{v}']\rangle\end{aligned} \tag{4.6}$$

and the angle brackets denote a spatial average with respect to the variable $\boldsymbol{x}$ over a cube of dimensions $L \times L \times L$. Although (4.5) is of interest in its own right, only the nonlinear terms are of interest here. The reason for focusing on the nonlinear terms will become clear in §§ 5 and 6.

Using the incompressibility condition $\nabla\cdot\boldsymbol{v} = 0$, integration by parts, and the fact that the surface integrals vanish in the limit $L \to \infty$ as shown in § 2, one finds

$$\langle \boldsymbol{v}'\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\rangle = \langle v'_i\partial_j(v_iv_j)\rangle = -\langle(\partial_jv'_i)(v_iv_j)\rangle = -\frac{\partial}{\partial r_j}\langle v'_iv_iv_j\rangle, \tag{4.7}$$

where $\partial_j = \partial/\partial x_j$, repeated indices are summed from 1 to 3, and the volume average is performed with respect to the variable $\boldsymbol{x}$. The simple fact

$$\frac{\partial f(\boldsymbol{x}+\boldsymbol{r})}{\partial x_j} = \frac{\partial f(\boldsymbol{x}+\boldsymbol{r})}{\partial r_j} \tag{4.8}$$

has also been used. By a similar procedure, one obtains

$$\langle \boldsymbol{v}\cdot(\boldsymbol{v}'\cdot\nabla)\boldsymbol{v}'\rangle = \langle v_i\partial_j(v'_iv'_j)\rangle = \frac{\partial}{\partial r_j}\langle v_iv'_iv'_j\rangle. \tag{4.9}$$

It is now shown that

$$\nabla_{\boldsymbol{r}} \cdot \langle|\delta \boldsymbol{v}(\boldsymbol{r})|^2 \delta \boldsymbol{v}(\boldsymbol{r})\rangle = 2 \frac{\partial}{\partial r_j} \langle v_i v_i' v_j - v_i' v_i v_j' \rangle, \tag{4.10}$$

where $\delta \boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x})$, and, therefore,

$$\langle \boldsymbol{v}' \cdot (\boldsymbol{v} \cdot \nabla)\boldsymbol{v} + \boldsymbol{v} \cdot (\boldsymbol{v}' \cdot \nabla)\boldsymbol{v}' \rangle = -\tfrac{1}{2} \nabla_{\boldsymbol{r}} \cdot \langle|\delta \boldsymbol{v}(\boldsymbol{r})|^2 \delta \boldsymbol{v}(\boldsymbol{r})\rangle. \tag{4.11}$$

By homogeneity, $\langle v_i v_i v_j \rangle = \langle v_i' v_i' v_j' \rangle$ and, therefore,

$$\langle|\delta \boldsymbol{v}(\boldsymbol{r})|^2 \delta \boldsymbol{v}(\boldsymbol{r})\rangle = \langle (v_i' - v_i)(v_i' - v_i)(v_j' - v_j) \rangle \tag{4.12}$$

$$= \langle v_i v_i v_j' \rangle - \langle v_i' v_i' v_j \rangle - 2\langle v_i' v_i v_j' \rangle + 2\langle v_i v_i' v_j \rangle. \tag{4.13}$$

Incompressibility implies that

$$\frac{\partial}{\partial r_j} \langle v_i v_i v_j' \rangle = \langle v_i v_i \partial_j v_j' \rangle = 0, \tag{4.14}$$

$$\frac{\partial}{\partial r_j} \langle v_i' v_i' v_j \rangle = \langle v_j \partial_j (v_i' v_i') \rangle = \langle \partial_j (v_i' v_i' v_j) \rangle = 0. \tag{4.15}$$

This proves (4.10) and (4.11).

Using the condition $\nabla \cdot \boldsymbol{b} = 0$ and integrating by parts, the remaining nonlinear terms in (4.6) are

$$-\langle \boldsymbol{v}' \cdot (\boldsymbol{b} \cdot \nabla)\boldsymbol{b} \rangle = -\langle v_i' \partial_j (b_i b_j) \rangle = \langle (\partial_j v_i')(b_i b_j) \rangle = \frac{\partial}{\partial r_j} \langle v_i' b_i b_j \rangle, \tag{4.16}$$

$$-\langle \boldsymbol{v} \cdot (\boldsymbol{b}' \cdot \nabla)\boldsymbol{b}' \rangle = -\langle v_i \partial_j (b_i' b_j') \rangle = -\frac{\partial}{\partial r_j} \langle v_i b_i' b_j' \rangle, \tag{4.17}$$

$$\langle \boldsymbol{b}' \cdot (\boldsymbol{v} \cdot \nabla)\boldsymbol{b} \rangle = \langle b_i' \partial_j (b_i v_j) \rangle = -\langle (\partial_j b_i')(b_i v_j) \rangle = -\frac{\partial}{\partial r_j} \langle b_i b_i' v_j \rangle, \tag{4.18}$$

$$-\langle \boldsymbol{b}' \cdot (\boldsymbol{b} \cdot \nabla)\boldsymbol{v} \rangle = -\langle b_i' \partial_j (v_i b_j) \rangle = \langle (\partial_j b_i')(v_i b_j) \rangle = \frac{\partial}{\partial r_j} \langle v_i b_i' b_j \rangle, \tag{4.19}$$

$$\langle \boldsymbol{b} \cdot (\boldsymbol{v}' \cdot \nabla)\boldsymbol{b}' \rangle = \langle b_i \partial_j (b_i' v_j') \rangle = \frac{\partial}{\partial r_j} \langle b_i b_i' v_j' \rangle, \tag{4.20}$$

$$-\langle \boldsymbol{b} \cdot (\boldsymbol{b}' \cdot \nabla)\boldsymbol{v}' \rangle = -\langle b_i \partial_j (v_i' b_j') \rangle = -\frac{\partial}{\partial r_j} \langle b_i b_j' v_i' \rangle. \tag{4.21}$$

By the same procedure as employed to derive (4.10), it can be shown that

$$\nabla_{\boldsymbol{r}} \cdot \langle|\delta \boldsymbol{b}|^2 \delta \boldsymbol{v}\rangle = 2 \frac{\partial}{\partial r_j} \langle b_i b_i' v_j - b_i' b_i v_j' \rangle, \tag{4.22}$$

where $\delta \boldsymbol{b}(\boldsymbol{r}) = \boldsymbol{b}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{b}(\boldsymbol{x})$ and

$$\nabla_{\boldsymbol{r}} \cdot \langle (\delta \boldsymbol{v} \cdot \delta \boldsymbol{b})\delta \boldsymbol{b} \rangle = -\frac{\partial}{\partial r_j} \langle v_i' b_i b_j' - v_i' b_i b_j + v_i b_i' b_j' - v_i b_i' b_j \rangle. \tag{4.23}$$

Thus, the sum of the terms (4.16)–(4.21) is equal to

$$\nabla_{\boldsymbol{r}} \cdot \left\langle (\delta \boldsymbol{v} \cdot \delta \boldsymbol{b})\delta \boldsymbol{b} - \tfrac{1}{2}|\delta \boldsymbol{b}|^2 \delta \boldsymbol{v} \right\rangle. \tag{4.24}$$

Adding this to (4.11), we find that the nonlinear term (4.6) can be written

$$NL = \nabla_{\boldsymbol{r}} \cdot \left\langle [\delta \boldsymbol{v}(\boldsymbol{r}) \cdot \delta \boldsymbol{b}(\boldsymbol{r})]\delta \boldsymbol{b}(\boldsymbol{r}) - \tfrac{1}{2}[|\delta \boldsymbol{v}(\boldsymbol{r})|^2 + |\delta \boldsymbol{b}(\boldsymbol{r})|^2]\delta \boldsymbol{v}(\boldsymbol{r}) \right\rangle. \tag{4.25}$$

Now define the quantity

$$\varepsilon(\boldsymbol{r}) = -\frac{1}{2}\frac{\partial}{\partial t}\langle \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}(\boldsymbol{x})\rangle|_{NL}, \tag{4.26}$$

where the subscript NL indicates the time rate of change due to the nonlinear terms in the MHD equations. The function $\varepsilon(\boldsymbol{r})$ is not constant and should not be confused with the cascade rate ε, although the two are related as shown in §§ 5 and 6. From (4.5) and (4.25) it follows that

$$\varepsilon(\boldsymbol{r}) = -\tfrac{1}{4}\nabla\cdot\boldsymbol{F}(\boldsymbol{r}) \tag{4.27}$$

where

$$\boldsymbol{F}(\boldsymbol{r}) = \langle[|\delta\boldsymbol{v}(\boldsymbol{r})|^2 + |\delta\boldsymbol{b}(\boldsymbol{r})|^2]\delta\boldsymbol{v}(\boldsymbol{r}) - 2[\delta\boldsymbol{v}(\boldsymbol{r})\cdot\delta\boldsymbol{b}(\boldsymbol{r})]\delta\boldsymbol{b}(\boldsymbol{r})\rangle \tag{4.28}$$

and $\delta\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x})$. The last three equations are the principal results of this subsection. The vector $\boldsymbol{F}$ is a statistical third-order moment constructed from the fluctuations $\delta\boldsymbol{v}$ and $\delta\boldsymbol{b}$. It is called the vector third-order moment for the energy.

4.2. *Third-order moment $\boldsymbol{F}^C$ for the cross-helicity*

To derive an equation for the two-point correlation function take the dot product of $\boldsymbol{b}'$ with (4.1), plus the dot product of $\boldsymbol{b}$ with (4.3), plus the dot product of $\boldsymbol{v}'$ with (4.2), plus the dot product of $\boldsymbol{v}$ with (4.4), and then average to obtain

$$\begin{aligned}\frac{\partial}{\partial t}\langle \boldsymbol{v}'\cdot\boldsymbol{b} + \boldsymbol{b}'\cdot\boldsymbol{v}\rangle + NL &= -\langle \boldsymbol{b}'\cdot\nabla p + \boldsymbol{b}\cdot\nabla p'\rangle + \nu\langle \boldsymbol{b}'\cdot\nabla^2\boldsymbol{v} + \boldsymbol{b}\cdot\nabla^2\boldsymbol{v}'\rangle \\ &\quad + \eta\langle \boldsymbol{v}'\cdot\nabla^2\boldsymbol{b} + \boldsymbol{v}\cdot\nabla^2\boldsymbol{b}'\rangle + \langle \boldsymbol{b}'\cdot\boldsymbol{f}_v + \boldsymbol{b}\cdot\boldsymbol{f}_v' + \boldsymbol{v}'\cdot\boldsymbol{f}_b + \boldsymbol{v}\cdot\boldsymbol{f}_b'\rangle,\end{aligned} \tag{4.29}$$

where

$$\begin{aligned}NL = \langle \boldsymbol{v}'\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{b} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{v}] + \boldsymbol{v}\cdot[(\boldsymbol{v}'\cdot\nabla)\boldsymbol{b}' - (\boldsymbol{b}'\cdot\nabla)\boldsymbol{v}'] \\ + \boldsymbol{b}'\cdot[(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - (\boldsymbol{b}\cdot\nabla)\boldsymbol{b}] + \boldsymbol{b}\cdot[(\boldsymbol{v}'\cdot\nabla)\boldsymbol{v}' - (\boldsymbol{b}'\cdot\nabla)\boldsymbol{b}']\rangle\end{aligned} \tag{4.30}$$

and the average is with respect to the variable $\boldsymbol{x}$ over a cube of dimensions $L\times L\times L$.

The nonlinear terms in (4.30) are evaluated in the same manner as described in § 4.1 with the results

$$\langle \boldsymbol{v}'\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{b}\rangle = \langle v_i'\partial_j(b_i v_j)\rangle = -\langle(\partial_j v_i')(b_i v_j)\rangle = -\frac{\partial}{\partial r_j}\langle v_i' b_i v_j\rangle, \tag{4.31}$$

$$-\langle \boldsymbol{v}'\cdot(\boldsymbol{b}\cdot\nabla)\boldsymbol{v}\rangle = -\langle v_i'\partial_j(v_i b_j)\rangle = \langle(\partial_j v_i')(v_i b_j)\rangle = \frac{\partial}{\partial r_j}\langle v_i' v_i b_j\rangle, \tag{4.32}$$

$$\langle \boldsymbol{v}\cdot(\boldsymbol{v}'\cdot\nabla)\boldsymbol{b}'\rangle = \langle v_i\partial_j(b_i' v_j')\rangle = \frac{\partial}{\partial r_j}\langle v_i b_i' v_j'\rangle, \tag{4.33}$$

$$-\langle \boldsymbol{v}\cdot(\boldsymbol{b}'\cdot\nabla)\boldsymbol{v}'\rangle = -\langle v_i\partial_j(v_i' b_j')\rangle = -\frac{\partial}{\partial r_j}\langle v_i v_i' b_j'\rangle, \tag{4.34}$$

$$\langle \boldsymbol{b}'\cdot(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\rangle = \langle b_i'\partial_j(v_i v_j)\rangle = -\langle(\partial_j b_i')(v_i v_j)\rangle = -\frac{\partial}{\partial r_j}\langle v_i b_i' v_j\rangle, \tag{4.35}$$

$$-\langle \boldsymbol{b}'\cdot(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}\rangle = -\langle b_i'\partial_j(b_i b_j)\rangle = \langle(\partial_j b_i')(b_i b_j)\rangle = \frac{\partial}{\partial r_j}\langle b_i' b_i b_j\rangle, \tag{4.36}$$

$$\langle \boldsymbol{b}\cdot(\boldsymbol{v}'\cdot\nabla)\boldsymbol{v}'\rangle = \langle b_i\partial_j(v_i' v_j')\rangle = \frac{\partial}{\partial r_j}\langle v_i' b_i v_j'\rangle, \tag{4.37}$$

$$-\langle \boldsymbol{b}\cdot(\boldsymbol{b}'\cdot\nabla)\boldsymbol{b}'\rangle = -\langle b_i\partial_j(b_i'b_j')\rangle = -\frac{\partial}{\partial r_j}\langle b_i'b_ib_j'\rangle. \tag{4.38}$$

The procedure used to derive (4.10), (4.22) and (4.23) can be applied to show that the sum of the terms (4.31)–(4.38) is equal to

$$NL = \tfrac{1}{2}\nabla_{\boldsymbol{r}}\cdot\langle[\,|\delta\boldsymbol{v}(\boldsymbol{r})|^2 + |\delta\boldsymbol{b}(\boldsymbol{r})|^2]\delta\boldsymbol{b}(\boldsymbol{r}) - 2[\delta\boldsymbol{v}(\boldsymbol{r})\cdot\delta\boldsymbol{b}(\boldsymbol{r})]\delta\boldsymbol{v}(\boldsymbol{r})\rangle. \tag{4.39}$$

With these results in hand, now define the quantity

$$\varepsilon_C(\boldsymbol{r}) = -\frac{1}{2}\frac{\partial}{\partial t}\langle \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}(\boldsymbol{x})\rangle|_{NL}, \tag{4.40}$$

where the subscript *NL* indicates the time rate of change due to the nonlinear terms in the MHD equations. The function $\varepsilon_C(\boldsymbol{r})$ is not constant and should not be confused with the cascade rate of cross-helicity ε_C, although the two quantities are closely related as shown in §§ 5 and 6. From (4.29) and (4.39), it follows that

$$\varepsilon_C(\boldsymbol{r}) = -\tfrac{1}{4}\nabla\cdot\boldsymbol{F}^C(\boldsymbol{r}), \tag{4.41}$$

where

$$\boldsymbol{F}^C(\boldsymbol{r}) = -\langle[\,|\delta\boldsymbol{v}(\boldsymbol{r})|^2 + |\delta\boldsymbol{b}(\boldsymbol{r})|^2]\delta\boldsymbol{b}(\boldsymbol{r}) - 2[\delta\boldsymbol{v}(\boldsymbol{r})\cdot\delta\boldsymbol{b}(\boldsymbol{r})]\delta\boldsymbol{v}(\boldsymbol{r})\rangle. \tag{4.42}$$

Note the similarity between (4.40)–(4.42) and (4.26)–(4.28). The quantity $\boldsymbol{F}^C$ is the vector third-order moment for the cross-helicity.

4.3. *Third-order moment $\boldsymbol{F}^M$ for the magnetic-helicity*

Write down the equations for the magnetic field and vector potential at two points $\boldsymbol{x}$ and $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{r}$:

$$\frac{\partial \boldsymbol{b}}{\partial t} = \nabla\times(\boldsymbol{v}\times\boldsymbol{b}) + \eta\nabla^2\boldsymbol{b} + \boldsymbol{f}_b, \tag{4.43}$$

$$\frac{\partial \boldsymbol{a}}{\partial t} = \boldsymbol{v}\times\boldsymbol{b} + \eta\nabla^2\boldsymbol{a} + \boldsymbol{f}_a; \tag{4.44}$$

$$\frac{\partial \boldsymbol{b}'}{\partial t} = \nabla\times(\boldsymbol{v}'\times\boldsymbol{b}') + \eta\nabla^2\boldsymbol{b}' + \boldsymbol{f}_b', \tag{4.45}$$

$$\frac{\partial \boldsymbol{a}'}{\partial t} = \boldsymbol{v}'\times\boldsymbol{b}' + \eta\nabla^2\boldsymbol{a}' + \boldsymbol{f}_a', \tag{4.46}$$

where $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x}, t)$, $\boldsymbol{b}' = \boldsymbol{b}(\boldsymbol{x}+\boldsymbol{r}, t)$, etc., and all the spatial derivatives are with respect to the variable $\boldsymbol{x}$. Take the dot product of $\boldsymbol{a}'$ with (4.43), plus the dot product of $\boldsymbol{b}'$ with (4.44), plus the dot product of $\boldsymbol{a}$ with (4.45), plus the dot product of $\boldsymbol{b}$ with (4.46), and then average to obtain

$$\frac{\partial}{\partial t}\langle \boldsymbol{a}\cdot\boldsymbol{b}' + \boldsymbol{a}'\cdot\boldsymbol{b}\rangle = NL + \eta\langle \boldsymbol{a}\cdot\nabla^2\boldsymbol{b}' + \boldsymbol{b}'\cdot\nabla^2\boldsymbol{a} + \boldsymbol{a}'\cdot\nabla^2\boldsymbol{b} + \boldsymbol{b}\cdot\nabla^2\boldsymbol{a}'\rangle + \langle \boldsymbol{a}\cdot\boldsymbol{f}_b' + \boldsymbol{b}'\cdot\boldsymbol{f}_a + \boldsymbol{a}'\cdot\boldsymbol{f}_b + \boldsymbol{b}\cdot\boldsymbol{f}_a'\rangle, \tag{4.47}$$

where

$$NL = \langle \boldsymbol{a}\cdot[\nabla\times(\boldsymbol{v}'\times\boldsymbol{b}')] + \boldsymbol{b}\cdot(\boldsymbol{v}'\times\boldsymbol{b}') + \boldsymbol{a}'\cdot[\nabla\times(\boldsymbol{v}\times\boldsymbol{b})] + \boldsymbol{b}'\cdot(\boldsymbol{v}\times\boldsymbol{b})\rangle. \tag{4.48}$$

By repeated use of the vector identity (2.17), equation (4.47) can be transformed into

$$\frac{\partial}{\partial t}\langle \boldsymbol{a}\cdot\boldsymbol{b}' + \boldsymbol{a}'\cdot\boldsymbol{b}\rangle = NL - 2\eta\langle \boldsymbol{j}\cdot\boldsymbol{b}' + \boldsymbol{j}'\cdot\boldsymbol{b}\rangle + 2\langle \boldsymbol{a}\cdot\boldsymbol{f}_b' + \boldsymbol{a}'\cdot\boldsymbol{f}_b\rangle \tag{4.49}$$

and (4.48) can be transformed into

$$NL = 2\langle \boldsymbol{b}\cdot(\boldsymbol{v}' \times \boldsymbol{b}') + \boldsymbol{b}'\cdot(\boldsymbol{v} \times \boldsymbol{b})\rangle \tag{4.50}$$

or, equivalently,

$$NL = -2\nabla_{\boldsymbol{r}}\cdot\langle \boldsymbol{a} \times (\boldsymbol{v}' \times \boldsymbol{b}') - \boldsymbol{a}' \times (\boldsymbol{v} \times \boldsymbol{b})\rangle. \tag{4.51}$$

Because the divergence operator ($\nabla_{\boldsymbol{r}}\cdot$) acts only on the primed variables, the equivalence of these two expressions follows from an application of the vector identity $\nabla\cdot(\boldsymbol{A}\times\boldsymbol{B}) = \boldsymbol{B}\cdot(\nabla\times\boldsymbol{A}) - \boldsymbol{A}\cdot(\nabla\times\boldsymbol{B})$ to (4.51) followed by an application of the identity (2.17).

Equations (4.49) and (4.51) yield the following expression for the time rate of change due to the nonlinear terms

$$\varepsilon^M(\boldsymbol{r}) \equiv -\frac{1}{2}\frac{\partial}{\partial t}\langle \boldsymbol{a}(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{a}(\boldsymbol{x})\rangle|_{NL} = -\nabla\cdot\boldsymbol{F}^M, \tag{4.52}$$

where

$$\boldsymbol{F}^M(\boldsymbol{r}) = \langle \boldsymbol{a}(\boldsymbol{x}+\boldsymbol{r}) \times [\boldsymbol{v}(\boldsymbol{x}) \times \boldsymbol{b}(\boldsymbol{x})] - \boldsymbol{a}(\boldsymbol{x}) \times [\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r}) \times \boldsymbol{b}(\boldsymbol{x}+\boldsymbol{r})]\rangle \tag{4.53}$$

is the vector third-order moment for the magnetic-helicity. It does not appear possible to express the vector $\boldsymbol{F}^M$ solely in terms of the fluctuations as in (4.25) or (4.39). However, it is possible to manipulate (4.50) to obtain

$$NL = 2\langle \boldsymbol{b}'\cdot(\boldsymbol{v}\times\boldsymbol{b} - \boldsymbol{v}'\times\boldsymbol{b})\rangle = -2\langle \boldsymbol{b}'\cdot(\delta\boldsymbol{v}\times\boldsymbol{b})\rangle = -2\langle \delta\boldsymbol{b}\cdot(\delta\boldsymbol{v}\times\boldsymbol{b})\rangle = 2\langle \boldsymbol{b}\cdot(\delta\boldsymbol{v}\times\delta\boldsymbol{b})\rangle. \tag{4.54}$$

Thus, (4.52) is equivalent to

$$\varepsilon^M(\boldsymbol{r}) = -\langle \boldsymbol{b}(\boldsymbol{x})\cdot[\delta\boldsymbol{v}(\boldsymbol{r})\times\delta\boldsymbol{b}(\boldsymbol{r})]\rangle. \tag{4.55}$$

The magnetic-helicity vector $\boldsymbol{F}^M$ has a distinctly different character from the energy vector $\boldsymbol{F}$ or the cross-helicity vector $\boldsymbol{F}^C$. The difference is revealed by the special relation (4.55).

5. Spectral transfer rates of energy, cross-helicity and magnetic-helicity

The fourth pillar upon which the results of this paper are founded may be stated as follows.

In homogeneous but not necessarily isotropic turbulence, the energy transfer rate Π_K from small to large wavenumbers is related to the vector third-order moment $\boldsymbol{F}$ by

$$\Pi_K = -\frac{1}{8\pi^2}\int \nabla\cdot\left[\frac{\boldsymbol{r}}{r^2}\nabla\cdot\boldsymbol{F}\right]\frac{\sin(Kr)}{r}\,\mathrm{d}^3\boldsymbol{r}, \tag{5.1}$$

the cross-helicity transfer rate Π_K^C from small to large wavenumbers is related to the vector third-order moment $\boldsymbol{F}_C$ by

$$\Pi_K^C = -\frac{1}{8\pi^2}\int \nabla\cdot\left[\frac{\boldsymbol{r}}{r^2}\nabla\cdot\boldsymbol{F}^C\right]\frac{\sin(Kr)}{r}\,\mathrm{d}^3\boldsymbol{r}, \tag{5.2}$$

and the magnetic-helicity transfer rate Π_K^M from small to large wavenumbers is is related to the vector third-order moment $\boldsymbol{F}_M$ by

$$\Pi_K^M = -\frac{1}{2\pi^2}\int \nabla\cdot\left[\frac{\boldsymbol{r}}{r^2}\nabla\cdot\boldsymbol{F}^M\right]\frac{\sin(Kr)}{r}\,\mathrm{d}^3\boldsymbol{r} \tag{5.3}$$

or, equivalently,

$$\Pi_K^M = -\frac{1}{2\pi^2}\int \nabla\cdot\left[\frac{\boldsymbol{r}}{r^2}\langle \boldsymbol{b}\cdot(\delta\boldsymbol{v}\times\delta\boldsymbol{b})\rangle\right]\frac{\sin(Kr)}{r}\,\mathrm{d}^3\boldsymbol{r}. \tag{5.4}$$

These expressions are derived in the following subsections.

5.1. *Expression for the energy transfer rate*

By definition,

$$\varepsilon(\boldsymbol{r}) = -\frac{1}{2}\frac{\partial}{\partial t}\langle \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}(\boldsymbol{x})\rangle\big|_{NL}, \tag{5.5}$$

where the subscript *NL* indicates the time rate of change due to the nonlinear terms in the MHD equations. Substitute the decomposition (3.3) into this equation and use the fact that for any two functions $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$, $\langle f_K^<(\boldsymbol{x})g_K^>(\boldsymbol{x})\rangle = 0$. Thus

$$\begin{aligned}\varepsilon(\boldsymbol{r}) = -\frac{1}{2}\frac{\partial}{\partial t}\langle &\boldsymbol{v}_K^<(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}_K^<(\boldsymbol{x}) + \boldsymbol{v}_K^>(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}_K^>(\boldsymbol{x})\\ &+ \boldsymbol{b}_K^<(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}_K^<(\boldsymbol{x}) + \boldsymbol{b}_K^>(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}_K^>(\boldsymbol{x})\rangle\big|_{NL}.\end{aligned} \tag{5.6}$$

If it can be shown that

$$\varepsilon_K^<(\boldsymbol{r}) = -\frac{1}{2}\frac{\partial}{\partial t}\langle \boldsymbol{v}_K^<(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}_K^<(\boldsymbol{x}) + \boldsymbol{b}_K^<(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}_K^<(\boldsymbol{x})\rangle\big|_{NL}, \tag{5.7}$$

$$\varepsilon_K^>(\boldsymbol{r}) = -\frac{1}{2}\frac{\partial}{\partial t}\langle \boldsymbol{v}_K^>(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}_K^>(\boldsymbol{x}) + \boldsymbol{b}_K^>(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}_K^>(\boldsymbol{x})\rangle\big|_{NL}, \tag{5.8}$$

then, by the definition of Π_K,

$$\Pi_K = -\frac{1}{2}\frac{\partial}{\partial t}\langle |\boldsymbol{v}_K^<(\boldsymbol{x})|^2 + |\boldsymbol{b}_K^<(\boldsymbol{x})|^2\rangle\big|_{NL} = \varepsilon_K^<(\boldsymbol{r}=0), \tag{5.9}$$

where the last equality on the right-hand side follows from (5.7). Using the Fourier representation

$$\varepsilon_K^<(\boldsymbol{r}) = \sum_{|\boldsymbol{k}|\leqslant K}\hat{\varepsilon}(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}), \tag{5.10}$$

where

$$\hat{\varepsilon}(\boldsymbol{k}) = \frac{1}{L^3}\int_{cube}\varepsilon(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}, \tag{5.11}$$

one obtains from (5.9) the expression

$$\Pi_K = \frac{1}{L^3}\sum_{|\boldsymbol{k}|\leqslant K}\int\varepsilon(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}. \tag{5.12}$$

In the limit as $L\to\infty$, this becomes

$$\Pi_K = \frac{1}{(2\pi)^3}\int_{|\boldsymbol{k}|\leqslant K}\mathrm{d}^3\boldsymbol{k}\int\varepsilon(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}, \tag{5.13}$$

where the $\boldsymbol{r}$-integration is now over all space. Interchanging the order of integration in (5.13), the integration over $\boldsymbol{k}$ is performed by choosing a coordinate system in which the k_z-axis is aligned with $\boldsymbol{r}$. This yields

$$\Pi_K = \frac{1}{2\pi^2}\int\frac{\sin(Kr) - Kr\cos(Kr)}{r^3}\varepsilon(\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}. \tag{5.14}$$

An integration by parts then yields the final form

$$\Pi_K = \frac{1}{2\pi^2}\int \nabla\cdot\left[\frac{\boldsymbol{r}\varepsilon(\boldsymbol{r})}{r^2}\right]\frac{\sin(Kr)}{r}\,\mathrm{d}^3\boldsymbol{r}. \tag{5.15}$$

Insert (4.27) to obtain (5.1).

To complete the derivation it is necessary only to verify (5.7) and (5.8). Substitute the Fourier series for $\boldsymbol{v}(\boldsymbol{x})$ and $\boldsymbol{b}(\boldsymbol{x})$ into (5.5) to obtain

$$\varepsilon(\boldsymbol{r}) = -\frac{1}{2}\sum_{\boldsymbol{k}}\frac{\partial}{\partial t}\langle|\hat{\boldsymbol{v}}(\boldsymbol{k},t)|^2 + |\hat{\boldsymbol{b}}(\boldsymbol{k},t)|^2\rangle_{NL}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}) \tag{5.16}$$

and substitute the Fourier series for $\boldsymbol{v}_K^<(\boldsymbol{x})$ and $\boldsymbol{b}_K^<(\boldsymbol{x})$ to obtain

$$-\frac{1}{2}\frac{\partial}{\partial t}\langle\boldsymbol{v}_K^<(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}_K^<(\boldsymbol{x}) + \boldsymbol{b}_K^<(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}_K^<(\boldsymbol{x})\rangle\big|_{NL} = -\frac{1}{2}\sum_{|\boldsymbol{k}|\leqslant K}\frac{\partial}{\partial t}\langle|\hat{\boldsymbol{v}}(\boldsymbol{k},t)|^2 + |\hat{\boldsymbol{b}}(\boldsymbol{k},t)|^2\rangle_{NL}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}). \tag{5.17}$$

From the definition of $\varepsilon_K^<(\boldsymbol{r})$, a comparison of (5.16) and (5.17) shows that (5.17) is equal to $\varepsilon_K^<(\boldsymbol{r})$. This proves (5.7). The proof of (5.8) is similar.

5.2. *Expression for the cross-helicity transfer rate*

By definition,

$$\varepsilon^C(\boldsymbol{r}) = -\frac{1}{2}\frac{\partial}{\partial t}\langle\boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}(\boldsymbol{x})\rangle\big|_{NL}, \tag{5.18}$$

where the subscript *NL* indicates the contribution to the time rate of change arising from the nonlinear terms in the MHD equations. Use the decomposition (3.3) to find

$$\varepsilon^C(\boldsymbol{r}) = -\frac{1}{2}\frac{\partial}{\partial t}\langle\boldsymbol{v}_K^<(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}_K^<(\boldsymbol{x}) + \boldsymbol{b}_K^<(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}_K^<(\boldsymbol{x}) + \boldsymbol{v}_K^>(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}_K^>(\boldsymbol{x}) + \boldsymbol{b}_K^>(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}_K^>(\boldsymbol{x})\rangle\big|_{NL}. \tag{5.19}$$

As in the previous subsection, it can be shown that

$$\varepsilon_K^{C<}(\boldsymbol{r}) = -\frac{1}{2}\frac{\partial}{\partial t}\langle\boldsymbol{v}_K^<(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}_K^<(\boldsymbol{x}) + \boldsymbol{b}_K^<(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}_K^<(\boldsymbol{x})\rangle\big|_{NL}, \tag{5.20}$$

$$\varepsilon_K^{C>}(\boldsymbol{r}) = -\frac{1}{2}\frac{\partial}{\partial t}\langle\boldsymbol{v}_K^>(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{b}_K^>(\boldsymbol{x}) + \boldsymbol{b}_K^>(\boldsymbol{x}+\boldsymbol{r})\cdot\boldsymbol{v}_K^>(\boldsymbol{x})\rangle\big|_{NL}. \tag{5.21}$$

Therefore, by the definition of Π_K^C,

$$\Pi_K^C = -\frac{\partial}{\partial t}\langle\boldsymbol{v}_K^<(\boldsymbol{x})\cdot\boldsymbol{b}_K^<(\boldsymbol{x})\rangle\big|_{NL} = \varepsilon_K^{C<}(\boldsymbol{r}=0), \tag{5.22}$$

where the last equality on the right-hand side follows from (5.20). The Fourier representation

$$\varepsilon_K^{C<}(\boldsymbol{r}) = \sum_{|\boldsymbol{k}|\leqslant K}\hat{\varepsilon}^C(\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}), \tag{5.23}$$

where

$$\hat{\varepsilon}^C(\boldsymbol{k}) = \frac{1}{L^3}\int_{cube}\varepsilon^C(\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,\mathrm{d}^3\boldsymbol{r}, \tag{5.24}$$

yields, from (5.22), the expression

$$\Pi_K^C = \frac{1}{L^3} \sum_{|\boldsymbol{k}| \leqslant K} \int_{cube} \varepsilon^C(\boldsymbol{r}) \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r}. \tag{5.25}$$

In the limit $L \to \infty$, this becomes

$$\Pi_K^C = \frac{1}{(2\pi)^3} \int_{|\boldsymbol{k}| \leqslant K} \mathrm{d}^3\boldsymbol{k} \int \varepsilon^C(\boldsymbol{r}) \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r}, \tag{5.26}$$

where the $\boldsymbol{r}$-integration is now over all space. Interchanging the order of integration in (5.26), the integration over $\boldsymbol{k}$ is then performed by choosing a coordinate system with the k_z-axis aligned with $\boldsymbol{r}$. This yields

$$\Pi_K^C = \frac{1}{2\pi^2} \int \frac{\sin(Kr) - Kr\cos(Kr)}{r^3} \varepsilon^C(\boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r}. \tag{5.27}$$

An integration by parts then yields the desired expression

$$\Pi_K^C = \frac{1}{2\pi^2} \int \nabla \cdot \left[\frac{\boldsymbol{r}\varepsilon^C(\boldsymbol{r})}{r^2} \right] \frac{\sin(Kr)}{r} \, \mathrm{d}^3\boldsymbol{r}. \tag{5.28}$$

Insert the result (4.41) to obtain (5.2).

5.3. *Expression for the magnetic-helicity transfer rate*

The derivation is nearly identical to that for the transfer rate of cross-helicity in § 5.2, but is presented in full for completeness. By definition,

$$\varepsilon^M(\boldsymbol{r}) = -\frac{1}{2} \frac{\partial}{\partial t} \langle \boldsymbol{a}(\boldsymbol{x} + \boldsymbol{r}) \cdot \boldsymbol{b}(\boldsymbol{x}) + \boldsymbol{b}(\boldsymbol{x} + \boldsymbol{r}) \cdot \boldsymbol{a}(\boldsymbol{x}) \rangle \big|_{NL}, \tag{5.29}$$

where the subscript *NL* indicates the contribution to the time rate of change arising from the nonlinear terms in the MHD equations. Applying the decomposition (3.3), this takes the form

$$\begin{aligned} \varepsilon^M(\boldsymbol{r}) = -\frac{1}{2} \frac{\partial}{\partial t} \langle & \boldsymbol{a}_K^<(\boldsymbol{x} + \boldsymbol{r}) \cdot \boldsymbol{b}_K^<(\boldsymbol{x}) + \boldsymbol{b}_K^<(\boldsymbol{x} + \boldsymbol{r}) \cdot \boldsymbol{a}_K^<(\boldsymbol{x}) \\ & + \boldsymbol{a}_K^>(\boldsymbol{x} + \boldsymbol{r}) \cdot \boldsymbol{b}_K^>(\boldsymbol{x}) + \boldsymbol{b}_K^>(\boldsymbol{x} + \boldsymbol{r}) \cdot \boldsymbol{a}_K^>(\boldsymbol{x}) \rangle \big|_{NL}. \end{aligned} \tag{5.30}$$

As demonstrated in § 5.1, it can be shown that

$$\varepsilon_K^{M<}(\boldsymbol{r}) = -\frac{1}{2} \frac{\partial}{\partial t} \langle \boldsymbol{a}_K^<(\boldsymbol{x} + \boldsymbol{r}) \cdot \boldsymbol{b}_K^<(\boldsymbol{x}) + \boldsymbol{b}_K^<(\boldsymbol{x} + \boldsymbol{r}) \cdot \boldsymbol{a}_K^<(\boldsymbol{x}) \rangle \big|_{NL}, \tag{5.31}$$

$$\varepsilon_K^{M>}(\boldsymbol{r}) = -\frac{1}{2} \frac{\partial}{\partial t} \langle \boldsymbol{a}_K^>(\boldsymbol{x} + \boldsymbol{r}) \cdot \boldsymbol{b}_K^>(\boldsymbol{x}) + \boldsymbol{b}_K^>(\boldsymbol{x} + \boldsymbol{r}) \cdot \boldsymbol{a}_K^>(\boldsymbol{x}) \rangle \big|_{NL}. \tag{5.32}$$

Therefore, from the definition of Π_K^M,

$$\Pi_K^M = -\frac{\partial}{\partial t} \langle \boldsymbol{a}_K^<(\boldsymbol{x}) \cdot \boldsymbol{b}_K^<(\boldsymbol{x}) \rangle \big|_{NL} = \varepsilon_K^{M<}(\boldsymbol{r} = 0), \tag{5.33}$$

where the last equality on the right-hand side follows from (5.31). The Fourier representation

$$\varepsilon_K^{M<}(\boldsymbol{r}) = \sum_{|\boldsymbol{k}| \leqslant K} \hat{\varepsilon}^M(\boldsymbol{k}) \exp(\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}), \tag{5.34}$$

where

$$\hat{\varepsilon}^M(\boldsymbol{k}) = \frac{1}{L^3} \int_{cube} \varepsilon^M(\boldsymbol{r}) \exp(-\mathrm{i}\boldsymbol{k} \cdot \boldsymbol{r}) \, \mathrm{d}^3\boldsymbol{r}, \tag{5.35}$$

yields, from (5.33), the expression

$$\Pi_K^M = \frac{1}{L^3} \sum_{|\boldsymbol{k}| \leqslant K} \int_{cube} \varepsilon^M(\boldsymbol{r}) \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\, \mathrm{d}^3\boldsymbol{r}. \tag{5.36}$$

In the limit as $L \to \infty$, this becomes

$$\Pi_K^M = \frac{1}{(2\pi)^3} \int_{|\boldsymbol{k}| \leqslant K} \mathrm{d}^3\boldsymbol{k} \int \varepsilon^M(\boldsymbol{r}) \exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\, \mathrm{d}^3\boldsymbol{r}, \tag{5.37}$$

where the $\boldsymbol{r}$-integration is now over all space. Interchanging the order of integration in (5.37), the integration over $\boldsymbol{k}$ is then performed by choosing a coordinate system with the k_z-axis aligned with $\boldsymbol{r}$. This yields

$$\Pi_K^M = \frac{1}{2\pi^2} \int \frac{\sin(Kr) - Kr\cos(Kr)}{r^3} \varepsilon^M(\boldsymbol{r})\, \mathrm{d}^3\boldsymbol{r}. \tag{5.38}$$

An integration by parts then yields the desired expression

$$\Pi_K^M = \frac{1}{2\pi^2} \int \nabla\cdot \left[\frac{\boldsymbol{r}\varepsilon^M(\boldsymbol{r})}{r^2}\right] \frac{\sin(Kr)}{r}\, \mathrm{d}^3\boldsymbol{r}. \tag{5.39}$$

Now insert the result (4.52) to obtain (5.3) and insert the result (4.55) to obtain (5.4).

6. Divergence laws for MHD turbulence

The fundamental laws for the third-order moments are now derived in the limit of large Reynolds numbers. These laws are valid for both stationary (steady state) turbulence and freely decaying turbulence.

6.1. *Law for the inertial range energy flux*

Under steady-state conditions, the average energy input due to forcing is balanced by the average energy dissipation due to viscous friction and resistive losses as described by (2.7). If the wavenumber spectra of the forcing functions $\boldsymbol{f}_v$ and $\boldsymbol{f}_b$ are confined to large scales near $\ell_0 \simeq K_0^{-1}$, and the kinetic and magnetic Reynolds numbers are very large, then both the viscous and resistive dissipation scales are very small compared to ℓ_0 and there exists an intermediate range of scales called the inertial range where the energy transfer rate from large to small scales is approximately constant. That is,

$$\Pi_K \simeq \varepsilon = \text{constant} \tag{6.1}$$

for all K in the inertial range. This argument can be made more precise. Hereafter, the value of the magnetic Prandtl number $Pr_m = \nu/\eta$ is assumed fixed (constant) since the theory does not depend on the precise numerical value of Pr_m.

In homogeneous but not necessarily isotropic turbulence, the vector third-order moment $\boldsymbol{F}(\boldsymbol{r})$ satisfies the divergence law

$$\nabla\cdot\boldsymbol{F} = -4\varepsilon \tag{6.2}$$

in the limit as $Re \to \infty$ and for r in the inertial range.

Assume that the wavenumber spectra of the forcing functions are zero beyond some cutoff wavenumber K_c or, otherwise, are negligible for $K \gg K_0$. Assume that for every value of the kinetic Reynolds number $Re \sim \nu^{-1}$, the MHD system approaches a statistically stationary state at large times characterized by finite values of the average energy, cross-helicity and magnetic-helicity. Also assume that as $Re \to \infty$ with

Pr_m = constant, the average dissipation rates of energy, cross-helicity and magnetic-helicity all converge to constant values ε, ε_C and ε_M.

At steady state, the energy cascade rate $\varepsilon(\nu)$ may depend on the inverse Reynolds number ν and is defined by

$$\langle \boldsymbol{v} \cdot \boldsymbol{f}_v + \boldsymbol{b} \cdot \boldsymbol{f}_b \rangle = \nu \langle |\boldsymbol{\omega}|^2 + Pr^{-1} |\boldsymbol{j}|^2 \rangle = \varepsilon(\nu), \tag{6.3}$$

where $\varepsilon(\nu) \to \varepsilon$ as $\nu \to 0$. The scale-by-scale energy budget (3.6) takes the form

$$\Pi_K = S_K - D_K. \tag{6.4}$$

Equation (3.8) can be written in the equivalent form

$$S_K = \langle \boldsymbol{v}_K^< \cdot \boldsymbol{f}_{vK}^< + \boldsymbol{b}_K^< \cdot \boldsymbol{f}_{bK}^< \rangle = \langle \boldsymbol{v} \cdot \boldsymbol{f}_{vK}^< + \boldsymbol{b} \cdot \boldsymbol{f}_{bK}^< \rangle, \tag{6.5}$$

because $\langle \boldsymbol{v}_K^> \cdot \boldsymbol{f}_{vK}^< \rangle = 0$ and $\langle \boldsymbol{b}_K^> \cdot \boldsymbol{f}_{vK}^< \rangle = 0$. The assumption that the forcing functions are confined to small wavenumbers implies

$$\boldsymbol{f}_{vK}^<(\boldsymbol{x}, t) \simeq \boldsymbol{f}_v(\boldsymbol{x}, t), \qquad \boldsymbol{f}_{bK}^<(\boldsymbol{x}, t) \simeq \boldsymbol{f}_b(\boldsymbol{x}, t) \qquad \text{for } K \gg K_0. \tag{6.6}$$

Hence,

$$S_K \simeq \langle \boldsymbol{v} \cdot \boldsymbol{f}_v + \boldsymbol{b} \cdot \boldsymbol{f}_b \rangle = \varepsilon(\nu) \qquad \text{for all } K \gg K_0. \tag{6.7}$$

It will be shown that for a fixed value of K,

$$\lim_{\nu \to 0} D_K = 0 \tag{6.8}$$

and, consequently, from (6.4), (6.7) and (6.3),

$$\lim_{\nu \to 0} \Pi_K = \varepsilon \qquad \text{for all } K \gg K_0. \tag{6.9}$$

This equation says that in the limit as $Re \to \infty$, the energy transfer rate from wavenumbers less than K to wavenumbers greater than K is constant, independent of K, for all K in the inertial range $K \gg K_0$ and, moreover, the value of the energy transfer rate is equal to the cascade rate ε. Thus, the intuitive result (6.1) has been given a precise mathematical formulation. To demonstrate (6.8) note that for any function $\phi(\boldsymbol{x}, t)$

$$\langle |\phi(\boldsymbol{x})|^2 \rangle = \sum_{\boldsymbol{k}} |\hat{\phi}(\boldsymbol{k})|^2, \tag{6.10}$$

$$\langle |\nabla \phi_K^<(\boldsymbol{x})|^2 \rangle = \sum_{|\boldsymbol{k}| \leqslant K} k^2 |\hat{\phi}(\boldsymbol{k})|^2 \leqslant K^2 \langle |\phi(\boldsymbol{x})|^2 \rangle. \tag{6.11}$$

Therefore, from (2.16) and (2.18),

$$D_K = \nu \langle |\boldsymbol{\omega}_K^<(\boldsymbol{x})|^2 \rangle + \eta \langle |\boldsymbol{j}_K^<(\boldsymbol{x})|^2 \rangle \leqslant \nu K^2 \langle |\boldsymbol{v}(\boldsymbol{x})|^2 + Pr^{-1} |\boldsymbol{b}(\boldsymbol{x})|^2 \rangle. \tag{6.12}$$

Equation (6.8) follows from the assumption that the average energy remains bounded as $\nu \to 0$.

Substitute the result $\Pi_K = \varepsilon$ into (5.1) to obtain

$$\Pi_K = \frac{1}{2\pi^2} \int f(\boldsymbol{r}) \frac{\sin(Kr)}{r^3} \, \mathrm{d}^3\boldsymbol{r} = \varepsilon \qquad \text{for all } K \gg K_0, \tag{6.13}$$

where

$$f(\boldsymbol{r}) = -\frac{r^2}{4} \nabla \cdot \left[\frac{\boldsymbol{r}}{r^2} \nabla \cdot \boldsymbol{F}(\boldsymbol{r}) \right]. \tag{6.14}$$

With the change of variable $\boldsymbol{r}' = K\boldsymbol{r}$, equation (6.13) takes the form

$$\Pi_K = \frac{1}{2\pi^2}\int f\left(\frac{\boldsymbol{r}}{K}\right)\frac{\sin(r)}{r^3}\,\mathrm{d}^3\boldsymbol{r} = \varepsilon, \qquad \text{for all } K \gg K_0. \tag{6.15}$$

For large K, this integral depends on the behaviour of $f(\boldsymbol{r})$ near $\boldsymbol{r}=0$. Approximating $f(\boldsymbol{r})$ by the first term in its Taylor series, the integration yields

$$f(\boldsymbol{r}) = \varepsilon, \qquad K_0 r \lll 1. \tag{6.16}$$

The substitution of (6.16) into (6.14) yields the differential equation

$$r^2\nabla\cdot\left[\frac{\boldsymbol{r}}{r^2}\nabla\cdot\boldsymbol{F}(\boldsymbol{r})\right] = -4\varepsilon \tag{6.17}$$

or

$$r^2\nabla\cdot\left[\varepsilon(\boldsymbol{r})\frac{\boldsymbol{r}}{r^2}\right] = \varepsilon, \tag{6.18}$$

where $\varepsilon(\boldsymbol{r})$ is defined by (4.27). The solution of this equation that remains bounded near $r=0$ is

$$\varepsilon(\boldsymbol{r}) = \varepsilon. \tag{6.19}$$

This is (6.2).

So far, it has been assumed that the turbulence is stationary in time. It will now be shown that the divergence law (6.2) also holds in the case of freely decaying turbulence. In this case, however, the divergence law applies only to the nonlinear phase of the decay when there exists a large separation of length scales and a large inertial range. In the late stages of the decay, the dissipation acts on all dynamically relevant scales so the approximation $D_K = 0$ fails to hold. For freely decaying turbulence, the cascade rate is defined by the energy equation

$$-\frac{1}{2}\frac{\partial}{\partial t}\langle|\boldsymbol{v}|^2 + |\boldsymbol{b}|^2\rangle = \nu\langle|\boldsymbol{\omega}|^2\rangle + \eta\langle|\boldsymbol{j}|^2\rangle = \varepsilon(t). \tag{6.20}$$

The scale-by-scale energy budget (3.6) takes the form

$$-\frac{\partial E_K}{\partial t} = \Pi_K + D_K. \tag{6.21}$$

In the nonlinear phase of the decay, the approximation $D_K = 0$ is valid as long as the inertial range is well separated from the dissipation scales or, in other words, as long as the dissipation scales are greater than K. In addition,

$$\boldsymbol{v}_K^<(\boldsymbol{x},t) \simeq \boldsymbol{v}(\boldsymbol{x},t), \qquad \boldsymbol{b}_K^<(\boldsymbol{x},t) \simeq \boldsymbol{b}(\boldsymbol{x},t) \qquad \text{for } K \gg K_0 \tag{6.22}$$

so that

$$\frac{\partial E_K}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}\langle|\boldsymbol{v}_K^<|^2 + |\boldsymbol{b}_K^<|^2\rangle \simeq \frac{1}{2}\frac{\partial}{\partial t}\langle|\boldsymbol{v}|^2 + |\boldsymbol{b}|^2\rangle = -\varepsilon(t). \tag{6.23}$$

Thus, $\Pi_K = \varepsilon(t)$ and the rest of the derivation, from (6.13) onward, remains the same.

6.2. *Law for the inertial range cross-helicity flux*

In homogeneous but not necessarily isotropic turbulence, the vector third-order moment for the cross-helicity $\boldsymbol{F}^C(\boldsymbol{r})$ satisfies the divergence law,

$$\nabla\cdot\boldsymbol{F}^C = -4\varepsilon_C, \tag{6.24}$$

in the limit $Re \to \infty$ and for r in the inertial range.

The derivation of this result is almost identical to that for the energy in § 6.1. To summarize, it can be shown that $\Pi_K^C = \varepsilon_C$ for all $K \gg K_0$, and this result can be substituted into (5.2) to obtain (6.24). The details are left to the reader. The only notable difference between the derivation of (6.2) and (6.24) is in the demonstration

$$\lim_{\nu \to 0} D_K^C = 0. \tag{6.25}$$

To prove this, note that for any two functions $f(\boldsymbol{x})$ and $g(\boldsymbol{x})$, the Schwarz inequality implies

$$|\langle \nabla f_K^< \cdot \nabla g_K^< \rangle| \leqslant \langle |\nabla f_K^<|^2 \rangle^{1/2} \langle |\nabla g_K^<|^2 \rangle^{1/2} \leqslant K^2 \langle |f|^2 \rangle^{1/2} \langle |g|^2 \rangle^{1/2}, \tag{6.26}$$

where the inequality on the right-hand side follows from (6.11). From (3.24)

$$\langle \boldsymbol{j}_K^< \cdot \boldsymbol{\omega}_K^< \rangle = \langle \nabla v_{xK}^< \cdot \nabla b_{xK}^< + \nabla v_{yK}^< \cdot \nabla b_{yK}^< + \nabla v_{zK}^< \cdot \nabla b_{zK}^< \rangle, \tag{6.27}$$

where $\boldsymbol{v} = (v_x, v_y, v_z)$ and, therefore,

$$|D_K^C| = |(\nu + \eta) \langle \boldsymbol{j}_K^< \cdot \boldsymbol{\omega}_K^< \rangle| \leqslant 3\nu (1 + Pr^{-1}) \langle |\boldsymbol{v}|^2 \rangle^{1/2} \langle |\boldsymbol{b}|^2 \rangle^{1/2}, \tag{6.28}$$

If the total energy is bounded as $\nu \to 0$, then $\langle |\boldsymbol{v}|^2 \rangle$ and $\langle |\boldsymbol{b}|^2 \rangle$ are both bounded and the result (6.25) follows.

6.3. *Law for the inertial range magnetic-helicity flux*

In homogeneous but not necessarily isotropic turbulence, the vector third-order moment for the magnetic helicity $\boldsymbol{F}^M(\boldsymbol{r})$ satisfies the divergence law

$$\nabla \cdot \boldsymbol{F}^M = -\varepsilon_M \tag{6.29}$$

in the limit $Re \to \infty$ and for r in the inertial range. In addition, the third-order moment satisfies

$$\langle \boldsymbol{b} \cdot (\delta \boldsymbol{v} \times \delta \boldsymbol{b}) \rangle = -\varepsilon_M. \tag{6.30}$$

The derivation of these results follows closely the derivations for the energy and cross-helicity in §§ 6.1 and 6.2. Briefly, it can be shown that $\Pi_K^M = \varepsilon_M$ for all $K \gg K_0$ and this result can be substituted into (5.3) and (5.4) to obtain (6.29) and (6.30), respectively. The details are omitted for brevity.

7. Conclusions

It has been shown that in the limit of large kinetic and magnetic Reynolds numbers, the vector third-order moments $\boldsymbol{F}(\boldsymbol{r})$, $\boldsymbol{F}^C(\boldsymbol{r})$, and $\boldsymbol{F}^M(\boldsymbol{r})$ for homogeneous, anisotropic, incompressible MHD turbulence satisfy the divergence laws

$$\nabla \cdot \boldsymbol{F} = -4\varepsilon, \tag{7.1}$$

$$\nabla \cdot \boldsymbol{F}^C = -4\varepsilon_C, \tag{7.2}$$

$$\nabla \cdot \boldsymbol{F}^M = -\varepsilon_M, \tag{7.3}$$

and, in addition,

$$\langle \boldsymbol{b}(\boldsymbol{x}) \cdot [\delta \boldsymbol{v}(\boldsymbol{r}) \times \delta \boldsymbol{b}(\boldsymbol{r})] \rangle = -\varepsilon_M, \tag{7.4}$$

where $\boldsymbol{F}$, $\boldsymbol{F}^C$ and $\boldsymbol{F}^M$ are defined by (4.28), (4.42) and (4.53), respectively, and $\delta \boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{v}(\boldsymbol{x} + \boldsymbol{r}) - \boldsymbol{v}(\boldsymbol{x})$. These laws are valid in the inertial range for both statistically stationary turbulence and for freely decaying turbulence under the conditions discussed in § 2.1. It is important to emphasize that these laws are valid for anisotropic turbulence as well as isotropic turbulence and, therefore, are fundamental for the study of plasma turbulence in nature where statistical isotropy is often not satisfied.

What is the physical meaning of the laws (7.1)–(7.3)? The divergence law (7.1) expresses the fact that energy is conserved during the energy cascade process. In other words, the divergence law (7.1) is equivalent to the statement $\Pi_K = \varepsilon$ for all K in the inertial range. For this reason, (7.1) is called the law for the inertial range energy flux. The proof consists of two parts. If $\Pi_K = \varepsilon =$ const for all $K \gg K_0$, then, as shown following (6.13), $\nabla \cdot \boldsymbol{F} = -4\varepsilon$. On the other hand, if $\nabla \cdot \boldsymbol{F} = -4\varepsilon$, then it follows readily from (5.1) that $\Pi_K = \varepsilon$. This completes the proof. Similarly, the divergence laws (7.2) and (7.3) are equivalent to the conservation of the fluxes of cross-helicity and magnetic helicity in the inertial range, respectively.

The theory indicates that the quantity $\langle \boldsymbol{b} \cdot (\delta \boldsymbol{v} \times \delta \boldsymbol{b}) \rangle$ has special significance. The result (7.4) shows that this third-order moment is constant in the inertial range, independent of the scale $\boldsymbol{r}$ of the fluctuations, and is equal to the inverse cascade rate of magnetic-helicity. It is remarkable that this quantity does not depend on the magnetic vector potential. Consequently, this formula provides a practical means of determining the cascade rate of magnetic-helicity from experimental data. The presence of a strong mean magnetic field $\bar{\boldsymbol{b}}$ yields the approximation $\langle \boldsymbol{b} \cdot (\delta \boldsymbol{v} \times \delta \boldsymbol{b}) \rangle \simeq \bar{\boldsymbol{b}} \cdot \langle \delta \boldsymbol{v} \times \delta \boldsymbol{b} \rangle$.

Similar to the four-thirds law (1.5) for hydrodynamic turbulence, the theory implies that for homogeneous and isotropic, but not necessarily mirror symmetric MHD turbulence, the vector third-order moments take the form

$$\boldsymbol{F}(\boldsymbol{r}) = -\tfrac{4}{3}\varepsilon \boldsymbol{r}, \quad \boldsymbol{F}^C(\boldsymbol{r}) = -\tfrac{4}{3}\varepsilon_C \boldsymbol{r}, \quad \boldsymbol{F}^M(\boldsymbol{r}) = -\tfrac{1}{3}\varepsilon_M \boldsymbol{r}. \tag{7.5}$$

The first two equations are called the four-thirds laws for homogeneous isotropic incompressible MHD turbulence. Note that the radial (longitudinal) components of the equations for $\boldsymbol{F}$ and $\boldsymbol{F}^C$ in (7.5) are equivalent to the laws derived by Politano & Pouquet (1998*a*)

$$\langle [|\delta \boldsymbol{v}(\boldsymbol{r})|^2 + |\delta \boldsymbol{b}(\boldsymbol{r})|^2]\delta v_\parallel(\boldsymbol{r})\rangle - 2\langle [\delta \boldsymbol{v}(\boldsymbol{r}) \cdot \delta \boldsymbol{b}(\boldsymbol{r})]\delta b_\parallel(\boldsymbol{r})\rangle = -\tfrac{4}{3}\varepsilon r, \tag{7.6}$$

$$-\langle [|\delta \boldsymbol{v}(\boldsymbol{r})|^2 + |\delta \boldsymbol{b}(\boldsymbol{r})|^2]\delta b_\parallel(\boldsymbol{r})\rangle + 2\langle [\delta \boldsymbol{v}(\boldsymbol{r}) \cdot \delta \boldsymbol{b}(\boldsymbol{r})]\delta v_\parallel(\boldsymbol{r})\rangle = -\tfrac{4}{3}\varepsilon_C r. \tag{7.7}$$

The radial (longitudinal) component of the equation for $\boldsymbol{F}^M$ in (7.5) is

$$\langle (\boldsymbol{a}' \cdot \boldsymbol{b})v_\parallel - (\boldsymbol{a}' \cdot \boldsymbol{v})b_\parallel - (\boldsymbol{a} \cdot \boldsymbol{b}')v'_\parallel + (\boldsymbol{a} \cdot \boldsymbol{v}')b'_\parallel \rangle = -\tfrac{1}{3}\varepsilon_M r, \tag{7.8}$$

where $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{x})$, $\boldsymbol{a}' = \boldsymbol{a}(\boldsymbol{x} + \boldsymbol{r})$, $v_\parallel$ is the component of $\boldsymbol{v}$ in the $\boldsymbol{r}$-direction, etc., and $\boldsymbol{x}$ is a dummy variable. The last two terms on the left-hand side of (7.8) are not found in (17) of Politano *et al.* (2003). Therefore, the result (7.8) derived here does not agree with that of Politano *et al.* (2003). One difference between the formulation of Politano *et al.* (2003) and the present work is the definition of the cascade rate ε_M; the value of ε_M defined in equation (13) of Politano *et al.* (2003) is half the value defined by (4.52). However, this does not resolve the discrepancy between (17) of Politano *et al.* (2003) and (7.8) above.

The discrepancy arises from the fact that, by construction, (4.52) is symmetric with respect to the interchange of $\boldsymbol{x}$ and $\boldsymbol{x}'$, whereas the expressions derived by Politano *et al.* (2003) appear to lack such symmetry. The symmetry in (4.53) implies $\boldsymbol{F}^M(-\boldsymbol{r}) = -\boldsymbol{F}^M(\boldsymbol{r})$. In the special case when the turbulence is isotropic, the statistical third-order moment $\boldsymbol{F}^M(\boldsymbol{r})$ is constrained by tensor analysis to have the form $\boldsymbol{F}^M(\boldsymbol{r}) = A\boldsymbol{r}$ so that the condition $\boldsymbol{F}^M(-\boldsymbol{r}) = -\boldsymbol{F}^M(\boldsymbol{r})$ necessarily holds. The last two terms in (7.8), like the second term in (4.53), are necessary if $\boldsymbol{F}^M(\boldsymbol{r})$ is to have the correct symmetry. However, when the different definitions of ε_M are taken into account, (17) of Politano *et al.* (2003) implies (7.8).

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REFERENCES

Antonia, R. A., Ould-Rouis, M., Anselmet, F. & Zhu, Y. 1997 Analogy between predictions of Kolmogorov and Yaglom. *J. Fluid Mech.* **332**, 395–409.

Biskamp, D. 2003 *Magnetohydrodynamic Turbulence*. Cambridge University Press.

Frisch, U. 1995 *Turbulence: The Legacy of A. N. Kolmogorov*. Cambridge University Press.

Hill, R. J. 1997 Applicability of Kolmogorov's and Monin's equations of turbulence. *J. Fluid Mech.* **353**, 67–81.

Kolmogorov, A. N. 1941 Dissipation of energy in locally isotropic turbulence. *Dokl. Akad. Nauk SSSR,* **32**, 16–18, trans. in *Proc. R. Soc. Lond.* A **434**, 15–17 (1991).

Matthaeus, W. H., Ghosh, S., Oughton, S. & Roberts, D. A. 1996*a* Anisotropic three-dimensional MHD turbulence. *J. Geophys. Res.* **101**, 7619–7630.

Matthaeus, W. H., Bieber, J. W. & Zank, G. P. 1996*b* Anisotropic turbulence in the solar wind. *Solar Wind Eight* (ed. D. Winterhalter, J. T. Gosling, S. R. Habbal, W. S. Kurth & M. Neugebauer). *AIP Conf. Proc.* vol. 382, pp. 256–259.

Meneguzzi, M., Frisch, U. & Pouquet, A. 1981 Helical and nonhelical turbulent dynamos. *Phys. Rev. Lett.* **47**, 1060–1064.

Monin, A. S. 1959 Theory of locally isotropic turbulence. *Dokl. Akad. Nauk. SSSR* **125**, 515–518.

Monin, A. S. & Yaglom, A. M. 1975 *Statistical Fluid Mechanics,* vol. 2. MIT Press.

Montgomery, D. & Turner, L. 1981 Anisotropic magnetohydrodynamic turbulence in a strong external magnetic field. *Phys. Fluids* **24**, 825–831.

Montgomery, D. & Turner, L. 1982 Two-and-a-half-dimensional magnetohydrodynamic turbulence. *Phys. Fluids* **25**, 345

Oughton, S., Priest, E. R. & Matthaeus, W. H. 1994 The influence of a mean magnetic field on three-dimensional magnetohydrodynamic turbulence. *J. Fluid Mech.* **280**, 95–117.

Podesta, J. J., Forman, M. A. & Smith, C. W. 2007 Anisotropic form of third-order moments and relationship to the cascade rate in axisymmetric magnetohydrodynamic turbulence. *Phys. Plasmas* **14**, 092305.

Politano, H. & Pouquet, A. 1998*a* Dynamical length scales for turbulent magnetized flows. *Geophys. Res. Lett.* **25**, 273–276.

Politano, H. & Pouquet, A. 1998*b* Von Kármán–Howarth equation for magnetohydrodynamics and its consequences on third-order longitudinal structure and correlation functions. *Phys. Rev.* E **57**, 21.

Politano, H., Gomez, T. & Pouquet, A. 2003 Von Kármán–Howarth relationship for helical magnetohydrodynamic flows. *Phys. Rev.* E **68**, 026315.

Shebalin, J. V., Matthaeus, W. H. & Montgomery, D. 1983 Anisotropy in MHD turbulence due to a mean magnetic field. *J. Plasma Phys.* **29**, 525–547.

Taylor, J. B. 1986 Relaxation and magnetic reconnection in plasmas. *Rev. Mod. Phys.* **58**, 741–763.